Gödel's Ontological Dream

Mathematical Realism, Temporal Unrealism, and Gödel's God

"Die Welt ist vernünftig."
— KURT GÖDEL, "My Philosophical Outlook"[1] (*c.* 1960)

SECTION 1. Gödel's Mathematical platonism or Realism.

> The real argument for objectivism is the following. We know many general propositions about natural numbers to be true (2 `plus 2 is 4, there are infinitely many prime numbers, etc.) and, for example, we believe that Goldbach's conjuncture makes sense and must be either true or false without there being any room for arbitrary convention. *Hence, there must be objective facts about natural numbers. But these objective facts must refer to objects that are different from physical objects because, among other thing they are unchangeable in time.*
> —Gödel, "*Some Basic Theorems on the Foundations of Mathematics and Their [Philosophical] Implications*" [2]

The above quotation is from Gödel's last public lecture, the 26th Josiah Gibbs lecture to the American Mathematical Association on the 25th of December 1951 in Providence, Rhode Island. The subject of Gödel's lecture was (philosophical) implications of the Incompleteness Theorems.[3]

The year Gödel was inaugurating his *decennium mirabile* in mathematical logic and set theory was the same year (1928) that the Cambridge mathematician G. H. Hardy had been invited to deliver the sixth Josiah Gibbs lecture. Hardy, a confirmed bachelor and a classical *platonist*, or mathematical realist, described the work of the mathematician, engaged in pure as opposed to applied mathematics, as that of an observant spectator:

> The function of a mathematician, then, is simply to observe the facts about his own intricate system of reality, that astonishingly beautiful complex of logical relations which forms the subject-matter of his science, as if he were an explorer looking at a distant range of mountains, and to record the results of his observations in a series of maps, each of which is a branch of pure mathematics. ... Among them there perhaps none quite so fascinating, with quite the astonishing contrasts of sharp outline and shade, as that which constitutes the theory of numbers.[4]

This view of mathematics accords well with ancient Greek meaning of "theoretical". The mathematician proves *theorems*, which comes through Euclid from the ancient Greek θεώρημα (theṓrēma) meaning "speculation, proposition to be proved"). The mathematician is the spectator who in the theatre of the mind views the abstract patterns from on high, far removed from the competitive struggles in the arena of life.

Gödel's *platonism*[5], his realism about mathematical objects, properties and truths, is often presented as diametrically opposed to *experimental mathematics*, which uses computation to discover mathematical properties and patterns. Alan Turing pioneered experimental mathematics. He investigated the Riemann Hypothesis to calculate the zeros of the zeta function $\zeta(s)$ on the critical line.[6] Turing's last official research publication "A method for the calculation of the zeta function" [1943] discusses an algorithm which is still in use today and is known as "Turing's Method".

In a lecture delivered to the Mathematical Association of America and the American Mathematical Society in Cambridge, Massachusetts, in December 1933, "The present situation in the foundations of mathematics" Gödel had expressed caution about platonism:

> The result of the preceding discussion is that our axioms, if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind, and which does not even produce the conviction that they are consistent.[7]

Strikingly similar to Hardy's view is Gödel's most frequently quoted statement of platonism, the supplementary note to a reprinting of "What Cantor's Continuum Problem?" [1947] in Martin Davis's *The Undecidable* [1965]:

> Despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as true. I don't see any reason why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. The set-theoretical paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics.[8]

The allure of platonism was recounted in an anecdote told by G. H. Hardy, who described his discovery, and collaboration with, Strinivasa Ramanujan, as "the one romantic incident I my life".[9]

<table>
<tr><td rowspan="6">"He [Ramanujan] could remember the idiosyncrasies of numbers in an almost uncanny way. It was Littlewood who said that every positive integer was one of Ramanujan's personal friends. I remember once going to see him when he was ill at Putney. I had ridden in taxicab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two [positive] cubes in two different ways."</td><td>1</td><td>1</td><td>7</td><td>343</td></tr>
<tr><td>2</td><td>8</td><td>8</td><td>512</td></tr>
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<tr><td>4</td><td>64</td><td>10</td><td>1000</td></tr>
<tr><td>5</td><td>125</td><td>11</td><td>1331</td></tr>
<tr><td>6</td><td>216</td><td>12</td><td>1728</td></tr>
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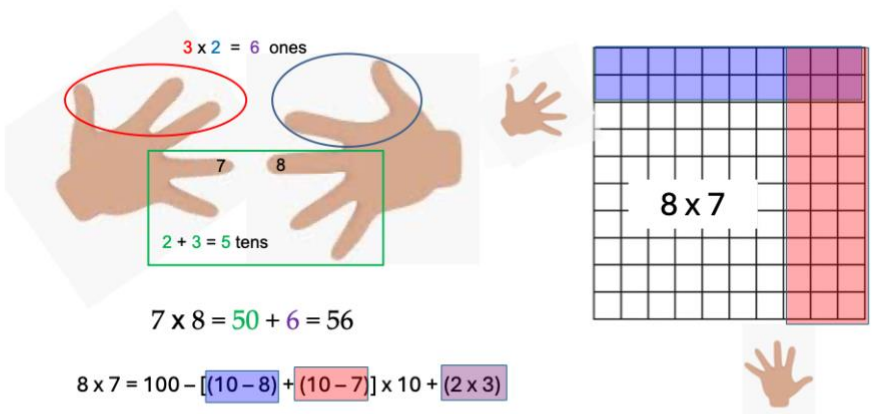
Puzzle #1. Hardy's anecdote supports a mystical, romanticized, account of platonism. How could a more mundane, but no less remarkable, computational explanation lead you to discover Ramanuma's taxicab number for yourself? Why is 1769 the smallest positive integer that can be expressed as the sum of two cubes in two different ways? (Solution in APPENDIX A).

Between the publication of his two most famous articles, Turing travelled to Princeton where he completed a dissertation under the direction of Alonzo Church. In this dissertation "Systems of Logic Based on Ordinals", Turing [1938] remarks:

> Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity. The activity of the intuition consists in making spontaneous judgements which are not the result of conscious trains of reasoning... The exercise of ingenuity in mathematics consists in aiding the intuition through suitable arrangements of propositions, and perhaps geometrical figures or drawings.[10]

The illustration below states an algorithm for using your fingers as a "digital computer" to calculate the products of the numbers from 6 to 10. Beginning with the little fingers and ending with the thumbs, label your fingers 6, 7, 8, 9, and 10. To multiply two numbers, simply *touch* the corresponding digits, e.g., 7 × 8 in the diagram above. The *multiplication algorithm* is simple: (1) *add* the fingers (touching and below) to compute the tens, in this case 2 + 3 = 5 tens or 50; (2) *multiply* the fingers above (the touching fingers on each hand) to calculate the ones, i.e., 3 × 2 = 6. Thus, 7 × 8 = 5 (tens) + 6 (ones), or 56.



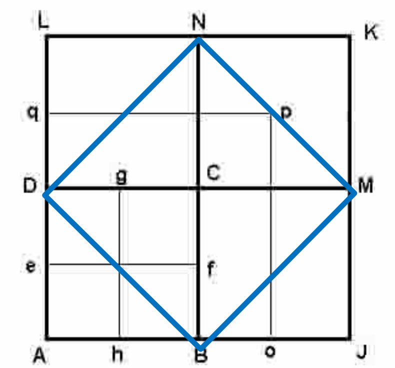
> Puzzle #2. Calculating or carrying out an algorithm by rote memory is different from *comprehending* why it works or *proving* that it works. Use Turing's idea of combining intuition and ingenuity to *explain* why the multiplication algorithm works. Rather than a computational proof that checks all possible products to verify the algorithm, use the interplay between arithmetic and geometry to explain *why* the algorithm works. Give an algebraic proof that the algorithm works. How could you generalize the algorithm to compute products in the upper decade from 16 – 20? (Solution in APPENDIX A).

The *locus classicus* for platonism is Plato's *Meno*. In this dialogue, Socrates's elicits a proof of the geometrical theorem from a slave boy untutored in mathematics. The dialogue is a philosophical classic because we read, and reread it, as a resource for discovering new insights into the nature of mathematics. One such insight is a digression, strictly speaking unnecessary, for the proof of the *Meno* theorem. This proof sets the stage for the first crisis in the foundations of mathematics.

When Socrates asks the slave boy "What is the length of the side of a square that *doubles* the area of the original square?" the boy, misled by language, answers that *doubling* the length of the side will double the area. This leads Socrates to draw a diagram, which reveals that doubling the side *quadruples* the area. At this point, Socrates could have immediately drawn the diamond square, whose side is the diagonal, and have asked questions to get the boy to see that the diagonal lines divide each of the four squares in half thus leading to a square whose area is half of 4 or 2.

However, this direct route to proving the theorem is not what transpires. Dividing the side of the original square in half produces smaller squares ¼ the size of the original square. The original square is now composed of 4 such squares. The square whose side is double that or the original square is composed of 16 quarter squares, which is four times the area of the original. This division of the original square in half allows for us to find better and better estimates. Consider the square whose side is 3 such units. Its area is $3 \times 3 = 9$, which is still larger than 8. This method of halving the units and finding better estimates can be iterated. In fact, this is the method used by the Babylonians (ad later by Newton) to calculate closer ad closer estimates of the length of the diagonal. This process of approximation never ends.

These calculations are not required for the geometrical proof of the theorem in the *Meno*. The inscribed (*blue*) diamond *doubles* the area of the original square. The diamond cuts each of the quadrants of the large square in half. Since the large square has 4 times the area of the original, the diamond has half of 4 or twice the area of the original. However, this digression, to paraphrase Hilbert's famous advice[11], contains the "germs of generality".



The unexamined assumption that geometrical ratios, are preserved under magnification is equivalent to assuming our underlying geometry is Euclidean. as opposed to two alternative non-Euclidean geometries. Labelling the units along the veridical axis 1, 2, 3, 4 we may plot a menagerie of new kinds of numbers, e.g., the diagonal's length $\sqrt{2} \approx 1.4142135623\ldots$ is *irrational*, as is the Golden ratio $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618033988 \ldots.$ Both these irrational numbers are *algebraic* but not *transcendental* as are the ubiquitous $\pi \approx 3.1415926 \ldots$ and Napier's constant

$$e \approx 2.7182818284 \ldots = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

The proof of the length of the diagonal is *irrational* (from the Greek *alogon*) is not a ratio of integral units of the sides can be derived from the simplest instance of the Pythagorean theorem $1^2 + 1^2 = d^2$ so $d = \sqrt{2}$. The classic proof is by *reductio ad absurdum*, and its logic is biased on Fermat's preferred form of mathematical induction, *disproof by infinite descent*.

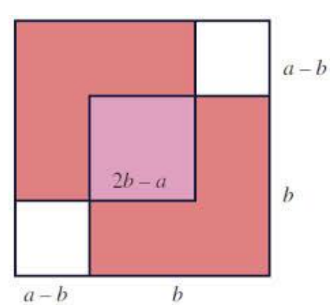
The geometric counterpart to proof by Fermat's disproof by infinite descent infinite is a fractal geometric construction anticipated by Stanley Tannenbaum (c.1950).[12] Incidentally, Tannenbaum arranged for John Conway, who later became the John von Neumann professor of mathematics at Princeton, to meet Gödel. Conway had discovered "surreal numbers" by generalizing Dedekind cuts which left him for weeks in a daze examining a previously unimagined realm of infinite and infinitesimal numbers.[13] Conway wanted to know what "God" [14] thought that this discovery was relevant to the first problem on Hilbert's list of open problems for 20th century mathematics, Cantor's Continuum Hypothesis.

Arithmetic Platonic Theorem

Prove: $\sqrt{2}$ is irrational.

1. Assume $\sqrt{2} = a/b$, with $(a, b) = 1$ and $b \neq 0$.
2. $2 = a^2 / b^2$, by squaring both sides
3. $2b^2 = a^2$, by multiplying both sides by b^2
4. Since $b \neq 0$, a is even and $a = 2c$ for some $c \neq 0$.
5. $2b^2 = (2c)^2$ by substitution lines 4 and 3.
6. Hence, $b^2 = 2c^2$, and so b is even.
7. Contradicting $(a, b) = 1$ since a and b are even,

Geometrical "Fractal" Proof

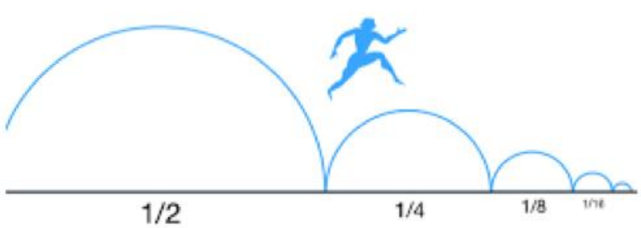


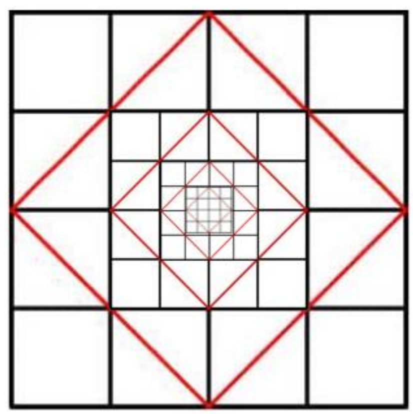
By *reductio ad absurdum* $\sqrt{2}$ is irrational. Q.E.D.

The classical theorem (*left*) proves the square root of 2 is irrational by *reductio ad absurdum*. The logic of this proof uses Fermat's disproof by infinite descent: if for any natural number *n*, if *n*'s having property F implies there is a natural number *m* less than *n* such that *m* has property F, then no natural number has the property F. The geometric proof (*right*) constructs an infinite regress in the form a fractal: if the original square is the sum of two squares, then there is a smaller square that is the sum of two squares, and so on *ad infinitum.*

Puzzle #3. How does the Tannenbaum diagram (*right*) prove that the diagonal of the square cannot be the sum of two squares by constructing an infinite regress or fractal?

Puzzle #4. Show that Fermat's disproof by infinite descent is logically equivalent to *strong induction*. *Fermat's disproof by infinite descent* states that if for any natural number *n* and for property F, if n's have F implies there is a natural number *m* less than *n* that has property F, then no natural number can have property F. Strong mathematical induction states that if for any natural number *n* if all natural numbers less than *n* having property F implies that *n* has property F, then all natural numbers have F.





Puzzle #5. The diagram (*left*[15]) is for Zeno's dichotomy: the runner who runs ½ the racecourse, then 1/4 more, then 1/8 more and so on *ad infinitum.* The diagram on the (*right*) iterates the Meno construction by inscribing a square within the diagram and a diamond with that square and so on *ad infinitum.* How can you "Zeno the *Meno*" by discovering Zeno's dichotomy within this fractal version of the *Meno*?

In this section, we have attempted to reconcile Gödel's mathematical platonism with Turing's computational "experimental mathematics", by using the latter as an epistemological "ladder" of discovery connecting mathematical ontology of Plato's "heaven" to earthly calculations that lead to the discovery of objectively true and timeless theorems. The interplay between arithmetic computation and geometric construction championed by Turing, opens the door to an empirically based explanation of the epistemology of mathematics.[16]

The puzzles included in this section were "experiments" meant to evoke the "*aha*!" moment of sudden illumination characteristic of mathematical discovery. After submitting what would be his last publication on mathematical logic, the *Dialectica* interpretation in 1957, Gödel, dissatisfied with the epistemology of platonism, turned the phenomenology of Edmund Husserl, whose experience of "sudden illumination" Gödel sought but did not find.

> At some time between 1906 and 1910 Husserl had a psychological crisis. He doubted whether he had accomplished anything and is wife was very sick. At some point in this period, everything suddenly became clear to Husserl, and he did arrive at some absolute knowledge. But one cannot transfer absolute knowledge to somebody else; therefore, one cannot publish it. A lecture on the nature of time also came during this period when Husserl's experience of seeing absolute knowledge took place. I myself have never had such an experience. For me, there is no absolute knowledge: everything goes only by probability. Both Descartes and Schelling explicitly reported an experience of sudden illumination when they began to see everything in a different light.[17]

SECTION 2. Gödel's Mathematical Mind v. Turing's Thinking Machines?

Gödel's Incompleteness Theorems [1931] are among the most profound results of 20th century mathematics. Popular accounts state that Gödel's Theorem shows "there are truths of arithmetic that cannot be proved". This statement is inaccurate on two counts. First, Gödel's theorem does not deal with provability in any *absolute* sense but only *relative* to a class of precisely defined formal systems. Secondly, there are *two* incompleteness *theorems*. We shall provide an elegant modal proof of the Second Incompleteness Theorem in the next section.

Here we discuss a philosophical consequence of the Second Incompleteness Theorem that appears to put Gödel and Turing in opposition to each other, at least insofar as their defenders have posed Gödel's platonistic view about the creativity of mathematical minds as diametrically opposed to Turing's mechanistic view of the brain. We shall argue that Turing's physicalism is not opposed to Gödel's platonism and that both views may indeed imply the falsity of materialism. These terms need to be defined.

Before doing so, we note that Alfred Tarski was keenly aware of the alleged incompatibility of the presumed physicalism of the Vienna Circle and his work on the semantic conception of truth and model theory. Seeking the endorsement of the Vienna Circle, Tarski [1944] intimated that his work on truth was consistent with physicalism.[18] Tarski flaunted the platonistic presuppositions of his research by taking a Latin maxim attributed to Aristotle from a passage in the *Nicomachean Ethics*—

Amicus Plato, sed magis amica veritas
("Where both are friends, it is right to prefer truth").

and toying with its inverse:

Inimicus Plato sed magis inimica falsitas
("Plato is an enemy[19], but falsehood is a greater enemy").

In the only first-person account of his university days attending the meetings of the Vienna Circle, Gödel described himself as a convinced platonist philosophically, but silently, opposed to the logical positivist conception of logic that conflated truth with proof (a kind of *verificationism*):

> The completeness theorem, mathematically, is indeed an almost trivial consequence of Skolem 1922 [*Some Remarks on Axiomatized Set Theory*]. However, the fact is that, at the time, nobody (including Skolem himself) drew this conclusion..... This blindness is indeed surprising. But I think the explanation is not hard to explain.... I may add that my objectivistic conception of mathematics and meta-mathematics in general, and of transfinite reasoning in particular, was fundamental also in my other work in logic. How indeed could one think of expressing metamathematics in the mathematical systems themselves, if the

latter are considered to consist of meaningless symbols which acquire some substitute meaning only through metamathematics?[20]

In his Josiah Willard Gibbs Lecture [1951], Gödel sets forth two implications of his Incompleteness Theorems. The first implication was the "inexhaustibility" of pure mathematics in its never-ending need for new axioms. The second implication is known as Gödel's dichotomy. "Either mathematics is incompleteable in this sense, that its axioms can never be comprised in a finite rule, that is to say, *the human mind (even within the realm of pure mathematics) infinitely surpassed the power of any finite machine, or else there exists absolutely unsolvable Diophantine problems…. "* (italics Gödel's). These "sharp results" were, according to Gödel, evidence of the falsity of materialism—limited only by the "underdeveloped state of philosophy".

1. Materialism implies the mind is mechanistic.
2. The Second Incompleteness Theorem implies the Inexhaustibility Thesis about mathematics.
3. If the Inexhaustibility Thesis is true, then Gödel's disjunction is true: either the mind is not mechanistic or there are absolutely unsolvable mathematical problems.
4. If rational optimism is warranted, then there are no absolutely unsolvable problems.
5. Therefore, assuming rational optimism, materialism is false.

Alan Turing has been portrayed as a broken-hearted adolescent who believed in souls but who became a mature materialist after the death of his 17-year-old friend and mentor Christopher Morcom. Alan Hodges in his biography *Alan Turing: the Enigma* [1983] paints a more accurate portrait. Turing was not a dogmatic materialist but an agnostic physicalist. In a handwritten note to Mrs. Morcom, two years after the death of Christopher, Alan was struggling to reconcile a scientific world view and his sense that "Chris is in some way alive now." While staying with the Morcom family, the teenaged Alan wrote "Nature of Spirit", an essay inspired by his reading of Eddington's *Science and the Physical World* [1928]. Alan was skeptical about materialistic determinism: "The conception then of being able to know the exact state of the universe then really must break down on the small scale. This means then that the theory which held that as eclipses etc. are pre-destined so were all our actions breaks down too." Then he speculates:

> As McTaggart shews matter is meaningless in the absence of spirit (throughout I do not mean by matter that which can be a solid a liquid or a gas so much as that which is dealt with by physics e.g. light & gravitation as well, i.e. that which forms the universe). Personally, I think that spirit is really eternally connected with matter but certainly not always by the same kind of body. I did believe it possible for a spirit at death to go to a universe entirely separate from our own, but I now consider that matter & spirit are so connected that this would be a contradiction in terms.[21]

In March 1954, three months before his death, Turing sent Robin Gandy four postcards entitled *"Messages from the Unseen World"* (alluding to Eddington's 1929 lecture *"Science and the Unseen World"*) in which he says he was thinking about how nature could produce non-computational outcomes. On one postcard Turing reports: "I'm trying to invent a new Quantum Mechanics, but it won't really work." Unfortunately, the world will never know what Turing would have found, for on June 7th, Turing died with a cyanide-laced apple with a bite taken out of it by his bedside. Whether Turing's death was a suicide or an accident is a matter of speculation.

Materialism can be defined as the view that ultimate reality is reducible to local interactions of material atoms. *Physicalism* can be defined as a commitment to the ontological entities presupposed by what is currently the best scientific theory of the physical world. Physicalism, therefore, is distinguishable from materialism. Chomsky has made the point that the mind/body problem exists not because minds are mysterious entities but rather that there is has been no consensus about the philosophy and physics of matter to replace the refuted picture of 17th century Cartesian kinematics. The distinction between materialism and physicalism leads to the possibility of a reconciliation.

Gödel's refutation of materialism is based on that premise that materialism implies a mechanistic view of the mind. Turing's physicalism, however, is not committed to materialism. In fact, in the sentence between stating his Inexhaustibility Thesis and the Gödel disjunction, Gödel states that it is possible that the brain, except for having a finite number of connections, might be a Turing Machine. B. Jack Copeland points out the inaccuracy of calling the view that brains are (essentially) Turing Machines the "Turing's Thesis" since Turing was actively investigating the possibility that brains are not deterministically mechanistic.[22] Prior to his death, Turing was investigating the consequences of quantum mechanics for computability: if quantum mechanics turns out to be the best scientific theory of the physical world and quantum mechanics, contrary to materialism, implies non-locality, then Turing's physicalism would also imply the falsity of materialism. Paradoxically, if the ontological commitments of quantum mechanics (e.g., quarks, twisters, or strings) turn out to be more like mathematical objects than material atoms, then there is the possibility that physicalism implies mathematical "platonism from below".

SECTION 3. Gödel Incompleteness in Modal Provability Logic.

> I have been again concerned with logic recently, using the methods that you so successfully applied for the proof of undecidable properties. Here I came to a result that seems remarkable to me. Namely, I was able to prove that the consistency of mathematics is unprovable.
>
> —JOHN VON NEUMANN'S letter to Gödel (Nov. 30, 1930)[23]

On September 7th, 1930, during the roundtable discussion on the foundations of mathematics that closed the Königsberg conference on the *Epistemology of the Exact Sciences*, Gödel quietly announced, in a meticulously crafted sentence, a concise statement of his First Incompleteness Theorem:

> Assuming the consistency of classical mathematics) one can even give examples of propositions (and in fact of those of the type of Goldbach or Fermat) that, while contentually true, are unprovable in the formal system of classical mathematics.[24]

Philosophers representing the leading philosophies of mathematics were present--Rudolf Carnap (logical positivism and Frege's logicism), Arend Heyting (Brouwer's intuitionism), John von Neumann (Hilbert's formalism), Friedrich Waismann (Wittgensteinian remarks) attended the Königsberg conference. However, Gödel's concise announcement of incompleteness fell on uncomprehending ears except for von Neumann, whose curiosity was piqued and who pressed for details in a private conversation with Gödel.

One of the great ironies in the history of mathematics was that at another conference in Königsberg on the very next day, David Hilbert gave his famous lecture triumphantly ending with the credo: for the mathematician there is "no *Ignorabimus.... Wir müssen wissen. Wir werden wissen!*" Hilbert not only professed his credo but proposed to limit the realm of mathematical knowledge to what could be verified by a "mechanical procedure."[25]

Several weeks after their discussion, von Neumann sent a letter (quoted above) to Gödel announcing his remarkable discovery that "the consistency of mathematics is unprovable." It must have been a great disappointment to von Neumann when Gödel informed him that thirteen days earlier, he had submitted for publication *Über formal unentscheidbare Sätze der* Principia Mathematica *und verwandter Systeme, I*, which already contained *Satz* XI, now known as Gödel's Second Incompleteness Theorem.

The irony was that this theorem shattered Hilbert's dream of giving the foundations of mathematics an absolutely reliable justification through a finitary proof of the consistency of mathematical reasoning.

Gödel informally explained his First Incompleteness Theorem noting that its analogy to the antinomy of the Liar (or other epistemological antinomies) "leaps to the eye."[26] Whereas the Liar sentence asserts of itself that it is *untrue*, the Gödel sentence says of itself that it is *unprovable* in a precisely specified formal system such as *Principia Mathematica* (PM). Supposing the Gödel sentence G to be provable in PM, then PM would be *inconsistent* insofar as it proves a sentence asserting its own unprovability. Suppose ~G to be provable in PM. Then G is unprovable in PM, assuming PM to be consistent. So, neither G nor its negation is provable in PM, *i.e.*, G is *undecidable* in PM. Therefore, assuming PM to be consistent, PM is *incomplete*.

> GÖDEL'S FIRST INCOMPLETENESS THEOREM: if a formal system is consistent and its axiom system has enough arithmetic so that its theorems can be listed by some mechanical procedure, then there exists an *undecidable* sentence in that formal system, which is therefore *incomplete*.[27]

Elegant proofs of Gödel's Second Incompleteness Theorem were discovered in *modal provability logics*, which emerged from the 1950s - 1970s. These logics were anticipated by Gödel's [1933*f*] *"An interpretation of intuitionistic propositional calculus."* Gödel's insight was that intuitionistic *truth* was characterized in terms of *proof*, which is a kind of *necessity*, and so modal axioms could formalize *properties of provability*:

(**T**)	$\Box P \to P$	*What is provable is true.*
(**K**)	$\Box(P \to Q) \to (\Box P \to \Box Q)$	*Whatever follows from what is provable is provable.*
(**4**)	$\Box P \to \Box\Box P$	*What is provable is provably provable.*

Leon Henkin [1952] asked the intriguing question whether the *positive* Gödelian sentence "I am *provable*" is provable. Martin Löb [1955] answered Henkin's question in the affirmative by showing that Peano Arithmetic proves Löb's Axiom:

(**L**) $\Box(\Box P \to P) \to \Box P$

A *Gödel modal probability logic* results from adding Löb's axioms (**L**), the Kripke axiom (**K**), all propositional tautologies, and the inference rules of *necessitation*

(**N**) From a proof of φ to infer $\Box\varphi$

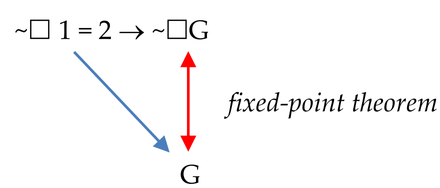
and *modus ponens*:

(**MP**) From $\varphi \to \psi$ and φ to infer ψ.

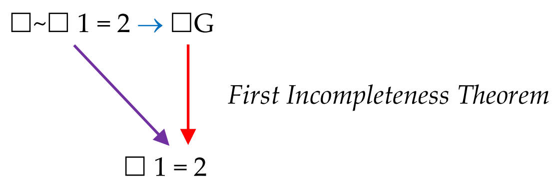
.

Howard de Jongh proved in 1975 that Axiom (**4**) was redundant by substituting $\Box\varphi \wedge \varphi$ for Gödel in (**L**).

Modal provability logic produces a succinct proof of Gödel's Second Incompleteness Theorem. The *fixed-point theorem* that yields the *Gödel sentence* such that $G \leftrightarrow \sim\Box G$. The consistency of Peano Arithmetic (PA) can be formally expressed by a CONS(PA): = $\sim\Box\ (1 = 2)$. The gist of the First Incompleteness Theorem is that the Gödel sentence is unprovable if CONS(PA), which we express by the horizontal conditional:



By the equivalence supplied by the fixed-point theorem and transitivity of implication, we have proved the diagonal implication. By the rule of necessitation, we prefix a □ and then distribute it over the implication by Kripke's axiom (**K**) to obtain the horizontal conditional:



By transitivity, we obtain the diagonal implication, which by contraposition, yields:

~□ 1 = 2 → ~□~□ 1 = 2.

Substituting the definition of CONS(PA) yields:

> **GÖDEL'S SECOND INCOMPLETENESS THEOREM**: CONS(PA) → ~□ CONS(PA), *i.e.*, if any formal system containing Peano Arithmetic is *consistent*, then that formal system cannot prove its own *consistency*.

In a lecture to a joint meeting of the Mathematical Association of America and the American Mathematical Society, Gödel summarized the significance his Incompleteness Theorems for Hilbert's program:

> The hope of finding a proof for freedom from contradiction undertaken by Hilbert and his disciples" had "vanished entirely in view of some recently discovered facts. It can be shown quite generally that there can exist no proof of the freedom of contradiction of a formal system S which could be expressed in terms of the formal system S itself …."[28]

SECTION 4. Gödel, Einstein, and the Unreality of Relativistic Time.

> Time is no specific character of being. In relativity theory the temporal relation is like far and near in space. I do not believe in the objectivity of time. The concept of *Now* never occurs in science itself, and science is supposed to be concerned with all that is objective.[29]

During the last fifteen years of his life at the Institute for Advanced Studies (IAS), Einstein sought the company of the younger Gödel (who was born the year after Einstein's *annus mirabilis* in 1905). Einstein told Morgenstern that he went to the Institute *'um das Privileg zu haben, mit Gödel zu Fuss nach Hause gehen zu durfen.'* A famous photo taken by Morgenstern shows Einstein, causally dressed in a rumbled sweatshirt with his wildly flowing hair, standing next to Gödel, dressed in a button-down double-breasted suit, impeccably groomed with every hair in place. The photo is a portrait of opposites.

To coax Gödel to publish so he could be advanced to full professor, John von Neumann had come up with the idea of asking Gödel to submit a paper to a *Festschrift* in honor of his friend Einstein's 70th Birthday. This led to Gödel's remarkable contributions to relativistic cosmology (Gödel [1949*a*,*b*,*c*]).

Einstein and Gödel, May 13, 1947
Photo: Oskar Morgenstern

Gödel's exploration into mathematical physics is perhaps surprising to those who know of him only as a logician. Gödel was a physics student at the university before switching to mathematics, and his interest in physics was maintained by talks with Einstein and by attending physics colloquia at IAS. After Gödel had agreed to contribute a short philosophical contribution for Einstein, Gödel came across a note by Gamow [1946] in *Nature* asking whether rotating cylindrical universes were consistent with general relativity. This convergence of questions stimulated Gödel to combine his *philosophical* defense of Kant's view on the non-objectivity of time (Gödel [1949a]) with his *mathematical* explorations into the cosmology of rotating Gödel universes in general relativity. Gödel's philosophical paper contained no mathematical equations but his technical paper [Gödel 1949] that appeared almost simultaneously in a special issue of the *Review of Modern Physics* did. Gödel discovered, and later elaborated upon (Gödel [1950]), a model in which the entire universe was rotating creating time-like loops making it possible to travel to the past by travelling into the future.[30]

Newton in his *Principia* [1687] had postulated the *metaphysis* of "absolute, true, mathematical time… [which] flows equably without relation to anything external," whereas Einstein in his Special Theory of Relativity [1905] adopted a *Machian* point of view, according to which time is something we abstract from *measurements* of time. Beginning with two principles:

[*Relativity*] The laws of physics are valid for all observers in all inertial frames of reference.
[*Constant c*] The vacuum speed of light is constant for all observers in all inertial frames of reference.

Einstein deduced that whether an observer measures two events to be happening "at the same time" depends on the observer's position and state of motion. This principle of the *Relativity of Simultaneity*, Einstein [1905] noted, undermines the privileged status of the present: "*The four-dimensional continuum is now no longer resolvable objectively into sections, all of which contain simultaneous events; 'now' loses for the spatially extended world its objective meaning.*"

Gödel noted, however, that Einstein's General Theory of Relativity [1915] allows for the reintroduction of a *universal cosmic time*. If the universe was non-expanding, Gödel noted, there are distinguished frames of reference which "follow the mean motion of matter" of the cosmos. Gödel explained in his lecture at IAS that his search for rotating solutions, or *Gödel universes*, was prompted by a desire to counter this objection in order to prove that time is *not objective* and so *unreal*:

> This incidentally also was the way in which I happened to arrive at the rotating solutions. I was working on the relationship… between Kant and relativistic physics insofar as in both theories the objective existence of a time in the Newtonian sense is denied. On this occasion one is led to observe that in the cosmological solutions known at present there does exist something like an absolute time…. So one is led to investigate whether or not this is a necessary property of all possible cosmological solutions.[31]

Gödel [1949*a*] linked relativity physics with the philosophical tradition of "Parmenides, Kant, and the modern idealists (*i.e.*, McTaggart) in the common denial of the "objective existence of … time in the Newtonian sense" (GCW-II, 202). McTaggart [1908] in setting forth his famous philosophical argument in "The Unreality of Time" enunciated various philosophical *dicta*, including the following: *"If one of the determinations past, present and future can ever be applied to [an event] then one of them has always been and always will be applicable, though of course not always the same one."*

There are various reasons for being interested in tense logic. Ordinary language is *tensed* whereas the language of physics is mathematical and *untensed*, and one can learn how to translate between the two types of expressions without confusing them. For example, the *connectedness* of time, the principle that "for any two distinct instants of time, one is earlier and other is later," can be expressed using modal temporal operators as *"whatever is going to have been the case either already has been or is now or will be."*

Using the modal temporal operators:

$\Box$	It *will always* be the case that
$\Diamond$	It *will* be the case that
$\blacksquare$	It *has always been* the case that
$\blacklozenge$	It *was once* the case that

McTaggart's statement yields temporal principles, here listed with their corresponding properties of temporal ordering:

$\blacklozenge P \to \blacksquare(\Diamond P \vee P \vee \blacklozenge P)$	*Connectedness of the past*
$\Diamond P \to \Box(\Diamond P \vee P \vee \blacklozenge P)$	*Connectedness of the future*
$\blacklozenge P \to \Box\blacklozenge P$	*Transitivity of "earlier than"*
$\Diamond P \to \blacksquare\Diamond P$	*Transitivity of "later than"*
$P \to \blacksquare\Diamond P$	*The past and future are converses*
$P \to \Box\blacklozenge P$	*The future and are converses*[32]

The third and fourth principles are instances of Axiom (**5**), and the last two principles are instances of the Brouwersche axiom. Intriguingly, both Axiom (**5**) and the Brouwersche axiom play key roles in contemporary explications of a Leibnizian modal ontological argument.

Consider the *tense logic*[33] axiomatized by:

Löb's Axiom:	$\blacksquare(\blacksquare P \to P) \to \blacksquare P$
McKinsey's Axiom:	$\Box\Diamond P \to \Diamond\Box P$

Löb's Axiom defines the *well-foundedness* of the past, *i.e.*, there is no infinite regress of moments of time into the past and the *transitivity* of "*later than*." On transitive frames, the McKinsey's Axiom defines the *atomicity* of time for the future, *i.e.*, for any moment of time, there is eventually a *maximal* moment of time. This tense logic characterizes the intuitive notion of time having no infinite regress into the past but proceeding in a linear sequence into the future.

Löb's Axiom implies the transitivity of *"earlier than"* and so time in this tense logic is *isotropic*: it is transitive independent of temporal direction. However, it is precisely this *transitivity* of time that leads to inconsistency with intuitive principles about time in these Gödel universes. The problem with Special Relativity is that there is *no uniquely objective way* of identifying the present moment. The problem with General Theory of Relativity is that it allows for the possibility of closed time-like loops rendering them *incompatible* with a tensed theory of time.[34] Hence, it is possible that time as characterized by the above tense logic does not exist.

Moreover, this modal characterization of temporality yields other intriguing results: (1) the corresponding relation for the McKinsey Axiom is *not first-order definable* and (2) the above logic is *modally incomplete* (*i.e.,* the tense logic holds in no first-order frame and yet is not inconsistent.)[35]

On March 15, 1951, Einstein handed out the First Einstein Award awarded jointly to Julian Schwinger and Gödel, saying to Schwinger "you deserve it"[36] and to Gödel "you don't need it." On this occasion, it was fitting that John von Neumann [1951], the first mathematician to grasp the revolutionary significance of Gödel's Incompleteness Theorems twenty-one years earlier, delivered a tribute to Gödel's work hailing it as "a landmark which will remain visible far in space and time."

Einstein called Gödel's curious cosmological gift "an important contribution to the general theory of relativity" but sought physical constraints for ruling out such universes.[37] Gödel himself drew a different conclusion: if time travel in terms of closed time-like loops is possible, then time itself is unreal. Time, like God, is either necessary or nothing; if it disappears in one possible universe, it is undermined in every possible universe, including our own. In other words, if it is *possible* that the existence of time is *impossible*, then time does *not* exist. This reasoning is an instance of the logically dual form of the Brouwersche Axiom,

(**B**◊) $\Diamond\Box{\sim}T \rightarrow {\sim}T$

which, as we shall see, plays a key role in Gödel's *Ontologischer Beweis*.

In his Incompleteness Theorems, Gödel affirmed the metaphysical reality of objective platonistic mathematics and drew conclusions about the incompleteness of the epistemological notion of proof within formal systems. However, in the case of relativistic time, Gödel affirmed the epistemological realism of relativity physics to draw a 'Kantian' conclusion about the metaphysical unreality, or non-objectivity, of time. The logical reasoning for this is, (or *may* have been) modal.

Special relativity implies the non-objectivity of A-series time since simultaneity is not absolute but relative to inertial frames. General Relativity implies the non-objectivity of B-series time, insofar as the axioms of temporal ordering depend upon the *contingent* distribution of matter in the cosmos. Unlike McTaggart's perverse, and philosophically frustrating, argument that assumes the logical dependence and incompatibility of A-series and B-series time, Gödel 's argument proceeds by cases to show that the non-objectivity of time is implied by both Special and General Relativity. Gödel's argument is not 'Kantian' in the sense of being an exposition of Kant's historical doctrines about the transcendental nature of space of time, but 'Kantian' in the sense of being "idealistic" or a philosophical argument against the objectivity of time.

> Another problem is the relation of our concept of time to real time. The real idea behind time is causation; the time of the structure of the world is just its causal structure. Causation in mathematics, in the sense of, say, a fundamental theorem causes its consequences, is not in time, but we take it as a scheme in time.[38]

Gödel's and Einstein's belief in the *unreality* of time perhaps was a defense against the *reality* of death. After failing to get a university position after graduate studies in physics, Einstein worked in obscurity in the patent office in Bern with Michele Besso, a friend since their years together at the Swiss Federal Polytechnic in Zurich. It was Besso who introduced Einstein to the positivism of Ernst Mach, which was instrumental in Einstein's discovery of special relativity.[39]

When Besso died, Einstein wrote in a letter dated 21 March 1955: "*Now he has departed from this strange world a little ahead of me. That signifies nothing. For those of us who believe in physics, the distinction between past, present and future is only a stubbornly persistent illusion.*" When it came time for his own death less the two weeks later, Einstein simply said, *"It's time to go."*

SECTION 6. Gödel's Ontological Deduction.

My belief is theistic; not pantheistic, following Leibniz rather than Spinoza. Spinoza's God is less than a person. Mine is more than a person….[40]

> Visited Gödel at the Institute today. A true miracle: he has gained 18 pounds, is in best shape, sparkling conversation. Lots about politics but then about his ontological proof—he had the result several years ago, is not happy with it but hesitates over publishing it. It would be concluded that he really believes in God, whereas he is only undertaking a logical investigation (*i.e.,* he shows that such a proof, appropriately axiomatized, is possible under classical assumptions—perfections, *etc.*). I joked that he should use a pseudonym—but he has already told Hempel and Scott about it, and I said people would recognize 'the claw of the lion', as they did with Newton."
>
> —OSKAR MORGENSTERN'S diary entry August 29, 1970[41]

Soon after changing his degree focus to mathematics, Gödel was invited to attend the meetings of the Vienna Circle. In these meetings, Gödel silently dissented from the Circle's presumed atheism and positivism. In his popularization of Logical Positivism, A. J. Ayer in *Language Truth and Logic* [1936] argued that "God exists" is not even false but *cognitively meaningless* because there could be no empirical experiences by which one could verify its truth or falsity. Gödel, however, regarded this atheism as a "prejudice of the times," and it was Gödel's *realism*—in contrast to the implicit *verificationism* of Brouwer's *intuitionism* and Hilbert's *formalism*—that led Gödel to distinguish between truth and proof and so discover his Completeness and Incompleteness Theorems.

Gödel, like Leibniz, worked on constructing a modal proof for God's existence. Gödel's handwritten notes for his *Ontologischer Beweis* occur in his notebooks ca. 1941 preceding his preoccupation with Leibniz from 1943–6. Gödel (completed but unsent) response to the Grandjean questionnaire[42] (quoted in the epigraph to this section) is the only autobiographical account of his early philosophical development.

However, it wasn't until February 1970 that Gödel, worrying about his eminent demise, decided to pass on his proof to Dana Scott.[43] Leibniz had criticized Descartes's ontological argument for having a logical gap: a failure to provide a proof of the possibility premise. Gödel bridged this gap by axiomatizing the notion of a positive property and then giving a *maximal consistency* argument to prove the possibility of the collection of all positive properties being instantiated. Gödel undoubtedly noticed the striking parallels between Leibniz's argument and his own non-constructive maximal consistency proof of the Completeness Theorem for first-order logic.[44]

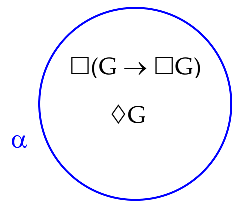
Here we present the *modal core* of Gödel's ontological discovery.[45] The argument requires an Anselmian premise (which contains the *converse* of the conditional in the antecedent of Löb's Axiom). If God exists, then God's existence would be a matter of *necessity*, or, in other words, God's existence could *not be accidental*. The Anselmian premise states that the above conditional itself is a necessary truth. The second premise asserts that God's existence is *possible*, for which Gödel provided the maximal consistency proof. The surprising conclusion from the above conceptually necessary truths is that God *actually* exists:

Anselmian Axiom:	$\Box(G \rightarrow \Box G)$
Possibility Premise:	$\Diamond G$
Actualist Conclusion:	$\therefore$ G

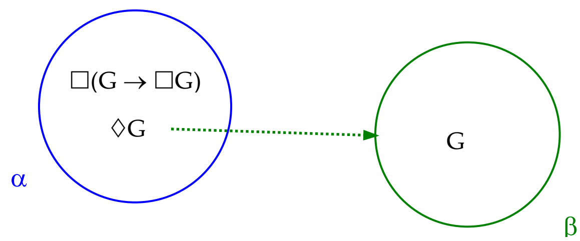
Using the Leibnizian idea that *necessity* is truth in *all* possible worlds and possibility is truth in *some* possible world, Kripke [1959] showed that the proliferation of syntactical characterizations of modality elegantly correspond to natural semantic conditions that could be required of the accessibility relation of "relative possibility" among a set of possible worlds. Although Axiom (5) is used in many contemporary versions of the modal ontological argument, the weaker Brouwersche Axiom is sufficient:

Brouwersche Axiom: $G \rightarrow \Box\Diamond G$ *Symmetry*

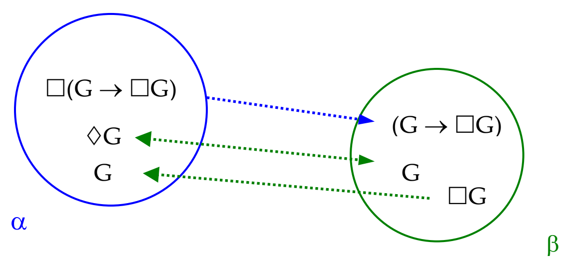
We sketch the modal core of Gödel's ontological argument using possible world diagrams. In the actual world α, the two modal premises are true:



If $\Diamond G$ is true in the actual world α, then in *some* world β possible relative to α, G is true:



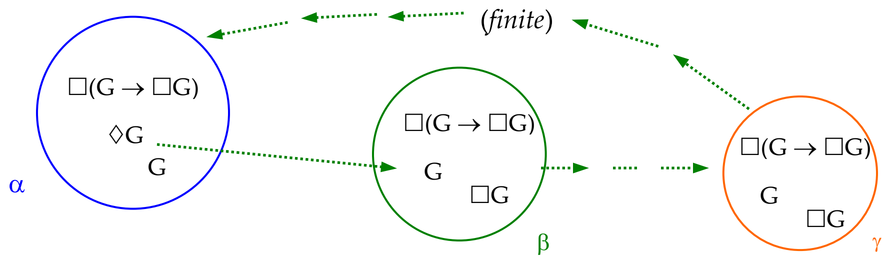
If $\Box(G \rightarrow \Box G)$ is true in α, then in *all* worlds possible relative to α, including β, $(G \rightarrow \Box G)$ is true. Hence, by *modus ponens* (since the laws of logic hold in all possible worlds) $\Box G$ is true in β. The Brouwersche Axiom makes the relative possibility relation *symmetric*, *i.e.,* all accessibility arrows between worlds are *double* arrows. Since $\Box G$ is true in β, assuming symmetry, G is true in α.



In fact, the Brouwersche Axiom allows the deduction of the *stronger* conclusion $\Box G$, namely that God's existence is *necessary*. A *weaker* condition suffices for a proof of God's *actual* existence:

$$\Diamond G \wedge \Box(G \rightarrow \Box G) \rightarrow G$$

namely, the principle of "finite return" [46], *i.e.,* from each successor world we can always return to α by a finite chain of successor worlds:



Some critics have accused Gödel of disingenuously distancing himself from his discovery as an *apologetic* argument, but here we have taken Gödel at his word and engaged in a *logical* investigation. However, as we shall see in the next section, conducting a logical investigation for Gödel did not preclude that investigation from having theological consequences.

SECTION 7. Gödel's Ontological Dream.

> "There is incomparably more knowable *a priori* than is currently known."
> —KURT GÖDEL, proposition 6, "My Philosophical Worldview" [*c*. 1960]

Gödel's articulated a rational vision of the logical and mathematical beauty of reality connecting more things in heaven and earth than dreamt of by other philosophers. This essay has tried to map a small constellation of these ideas.

Despite their technical sophistication, Gödel's theorems were not limited to mathematics but shed light on larger philosophical issues. How is this possible? One reason for the wide-ranging applicability of his theoretical results consists not only in Gödel's judicious choice of foundational research problems but in his technical virtuosity at *mathematizing* philosophical problems.

> The significance of mathematical logic for philosophy lies in its power to make thoughts explicit by illustrating the providing a frame for the axiomatic method. Mathematical logic makes explicit the central place of predication in the philosophical foundation of rational thought.[47]

> The fundamental principles are concerned with what the primitive concepts are and also their relationship. The axiomatic method goes step by step. We continue to discover new axioms; the process never finishes. Leibniz used formal analogy: in analogy with the seven stars in the Great Bear constellations, there are seven concepts. One should extend the analogy to cover the fact that by using the telescope we [now] see more stars in the constellation. [48]

Like the North Star in Ursa Major was used for navigation for centuries, Gödel's theorems have been a guiding light for philosophical exploration: he produced elegant theorems not only in mathematical logic, but also results approaching the status of theorems about the creativity of the mind and mechanistic calculation, about the modal status time in relativity physics, and a deduction of a God with all perfections (or positive properties) in theology. The seven points in Gödel's constellation explicated in this essay are:

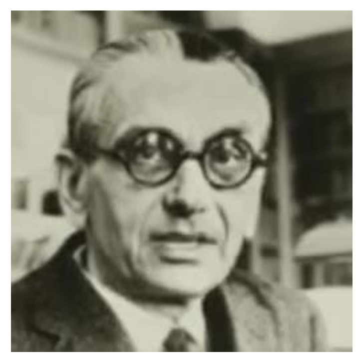
1. “The world is rational.”
2. “There is incomparably more knowable *a priori* than is currently known.”
3. If (1) and (2), then logic, mathematics, and physics reveal the rational structure of the world.
4. Mechanistic formal systems cannot be the cause of mathematical rationality because mathematical logic can be used to prove the limitations of such formal systems.
5. All mathematical truth cannot be captured by mechanistic formal systems (a philosophical implication of the Incompleteness Theorems).
6. If Einstein’s relativistic physics reveals the nature of physical reality, then (physical) time is not objectively real.
7. The only sufficient reason, or First Cause, of mathematics, logic, and physics is the mind of God; hence, there necessarily exists a unique, god-like being.

Remark (Theology): The reflection: according to the Principle of Sufficient Reason the world must have a cause. This must be necessary in itself (otherwise it would require a further cause). Proof of the existence of an *a priori* proof of the existence of God (the proof it contains fails to be one).[49]

Rationalism is connected with Platonism because it is directed to the conceptual aspect rather than toward the real [physical] world. One uses inductive evidence. It is surprising that in some parts of mathematics we get complete developments (such as the work by Gauss in number theory). Mathematics has a form of perfection. In mathematics one attains knowledge once for all. We may expect that the conceptual world is perfect and furthermore, that objective reality is beautiful, good, and perfect.[50]

“My Philosophical Outlook”
(c. 1960)

In principle, we can know all mathematics. It is given to us in its entirety and does not change—unlike the Milky Way. That part of it of which we have a perfect view seems beautiful suggesting harmony; that is that all the parts fit together although we see fragments of them only. Inductive inference is not like mathematical reasoning: it is based on equality or uniformity. But mathematics is applied to the real world and has proved fruitful. This suggests that the mathematical and empirical parts are in harmony and that real world is also beautiful. Otherwise, mathematics would be just an ornament, and the real world would be like an ugly body in beautiful clothing.[51]



KURT GÖDEL
(April 28, 1906 – January 14, 1978)

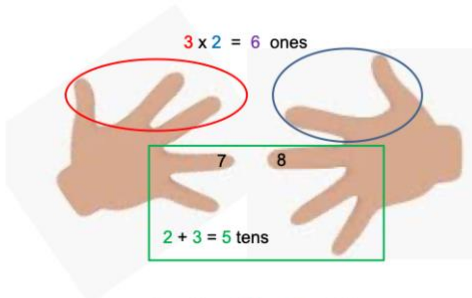
If the world [Welt] is rationally constructed and has meaning, then …”[52]

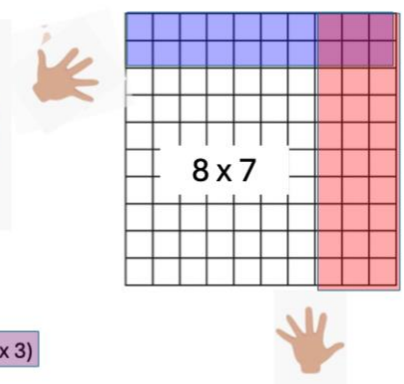
Beginning with this supposition, Gödel created a legacy of a set of theorems—in mathematics, physics, and theology— illuminating the philosophical outlook that the world is rational, good, and beautiful.

Appendix A. Solutions to Puzzles

Puzzle #1. Ramanujan-Hardy *taxicab numbers* can be expressed as a sum of two positive integer cubes in n distinct ways. The table of cubes has near coincidence: $9^3 = 729$ and $12^3 = 1728$. The numbers differ by 1 and 100, which are themselves cubes, $10^3 = 1000$ and $1^3 = 1$. Hence, $1729 = 9^3 + 10^3 = 1^3 + 12^3$.

Puzzle #2. The white 8 x 7 area is obtained by subtracting the lavender and salmon-colored areas but since the overlapping area has been subtracted twice, it needs to be added back.





$7 \times 8 = 50 + 6 = 56$

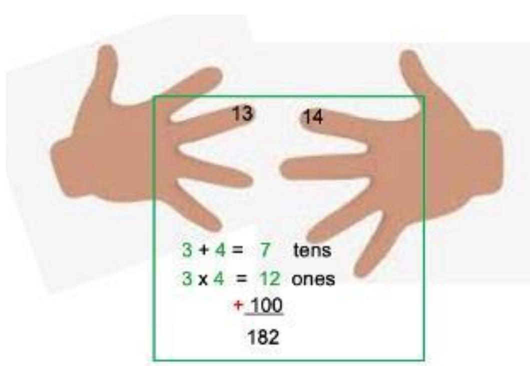
$8 \times 7 = 100 - [(10-8) + (10-7)] \times 10 + (2 \times 3)$

The general algebraic proof follows by substituting variables X and Y for 7 and 8.

$$\begin{array}{lcl} [(7-5)+(8-5)] \times 10 & = & (2+3) \times 10 = 50 \\ (10-7) \times (10-8) & = & 2 \times 3 = 6 \\ & & 56 \end{array}$$

$$\begin{array}{lcl} [(X-5)+(Y-5)] \times 10 & = & -100 + 10X + 10Y \\ (10-X) \times (10-Y) & = & \underline{100 - 10X - 10Y + XY} \\ & & XY \end{array}$$

To generalize: *add* the fingers touching and below to calculate the tens, *multiply* them to calculate the ones and then add 100.



$13 \times 14 = 70 + 12 + 100 = 182$

Puzzle #3. The Tannenbaum diagram proves the irrationality of the diagonal by a geometric infinite regress. The area of the square with side a is equal to two squares with side b. Place the two squares with side b inside the square with side a. Since the areas of the two b squares are equal to the big a square, by assumption, the light pink square in the center created by the overlapping b squares must be equal in area to the two smaller white squares, which are the remainders uncovered by the overlapping b squares. Notice that the pink square must be equal to the areas of the two white squares, and so on *ad infinitum*. This diagram can be compared with fractals illustrating Newton's method for calculation other irrational roots.

Conway and Shipman [2014] contains other diagrams to prove the irrationality of $\sqrt{3}$ and $\sqrt{5}$, using a a triangle and a pentagon, respectively. Can your find a proof of the irrationality of $\sqrt{5}$ by an infinite regress in the pentagon?

Puzzle #4. The equivalence of Strong Induction and Fermat's Disproof by Infinite Descent can be made perspicuous by formal logic (The theorem numbers are from Kalish, Montague, and Mar [2000], which can be consulted for details):

TS120 $\forall x[\mathrm{N}(x) \wedge \forall y[\mathrm{N}(y) \wedge y < x \rightarrow \mathrm{F}(y)] \rightarrow \mathrm{F}(x)] \rightarrow \forall x[\mathrm{N}(x) \rightarrow \mathrm{F}(x)]$

TS120.1 $\forall x[\mathrm{N}(x) \wedge ({\sim}\mathrm{F}(x) \rightarrow \exists y[\mathrm{N}(y) \wedge y < x \wedge {\sim}\mathrm{F}(y)]) \rightarrow \forall x[\mathrm{N}(x) \rightarrow {\sim}\mathrm{F}(x)]$

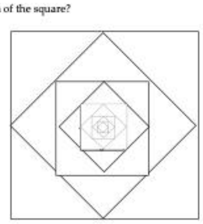
The idea for the derivation is simple: (1) substitute ~F for F in TS120; (2) exchange the second conjunct (*blue*) in the antecedent of TS120 with its logically equivalent contrapositive; (3) apply quantifier negation to the negated universal to obtain the existential consequent of the conditional in the antecedent of TS120.1.

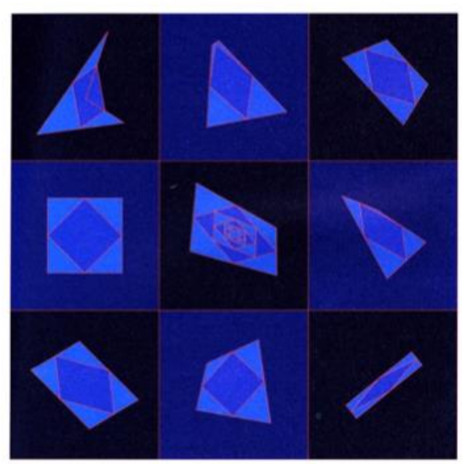
Puzzle #5. The *Meno* construction inscribes a diamond with half the area of the surrounding square; hence iterating the construction by inscribing a square within the diamond *ad infinitum* generates the geometric series of Zeno's dichotomy:

$$\sum_{1}^{\infty} {}^{1}\!/_{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

ZENO: Tell me, Socrates, can one not only *double* but also *half* the area of the square?
SOCRATES: Certainly.
ZENO: Then can we not *inscribe* a diamond within the unit square?
SocrAtes: How so?
ZENO: Can we not inscribe a square whose corners are the *midpoints* of each of the sides?
SOCRATES: Most certainly, Zeno.
ZENO: And what is the area of this diamond?
SOCRATES: I don't know.
ZENO: Can I divide the larger square into four smaller ones?
SOCRATES: Certainly.
ZENO: How much of the area of the larger square is each of the smaller squares?
SOCRATES: One-fourth.
ZENO. And the diagonal lines of the inscribed diamond, do they not divide each of the four smaller square in half?
SOCRATES: Certainly, and so their sum is four times one-half of one-fourth the area of the large square, or a twice the area of two of the smaller squares.
ZENO: And so?
SOCRATES: The diamond is half the area of the large square.
ZeNo: Excellent, Socrates! And this construct be repeated?
SOCRATES: How so?
ZENO: By constructing an inscribed square within the diamond whose corners are the midpoints of the diamond.
SOCRATES: Yes, this construct can be repeated indefinitely.
ZENO: The area of the inscribed figures?
SOCRATES: They will be half the area of the figure in which they are inscribed.
ZENO: And so is not the total area equal to:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots .$$





The Zeno construction and geometric series is universal: it holds for any quadrilateral. This illustration is from Eli Maor and Eugen Jost's *Beautiful Geometry*, Princeton University Press, 2017, 10.

APPENDIX B. Gödel's GOD

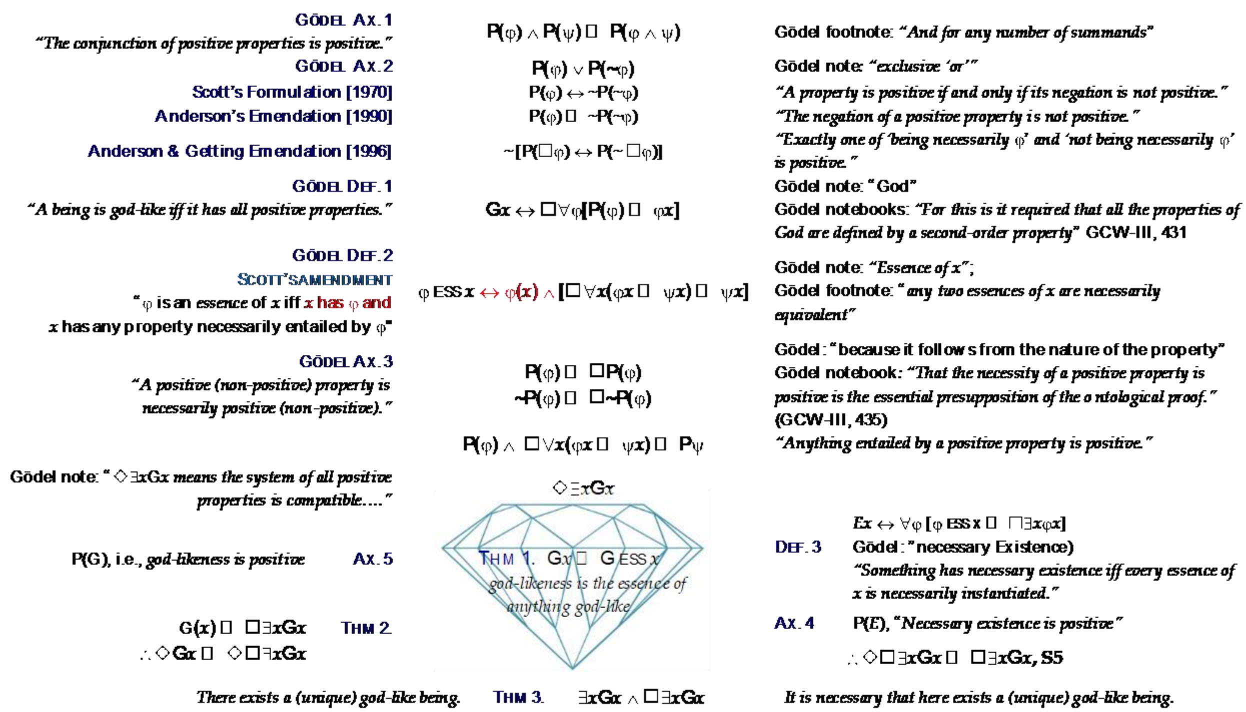
There are three stages to Gödel's Ontological Deduction (GOD):

(1) a MAXIMAL CONSISTENCY PROOF of the set of all perfections or positive properties akin to Henkin's version of the Gödel Completeness Theorem (which is how the theorem is typically proven today);

(2) the MODAL CORE as discussed in the essay, which can be derived using the Brouwersche (**B**) rather than the stronger (**5**) axiom; and

(3) the DEFINITIONS and THEOREMS (*blue font*) allow for the derivation of divine attributes such as necessary existence from the definition of god-likeness. Gödel's notes are in *italics.*

Christoph Benzmüller *et al.* [2013] computationally verified Gödel's deduction and subsequently claimed a "success story for AI" [2016] using an interactive human-AI interface to produce a model to show Gödel's version of God to be "inconsistent"; however, the conjunction (*red*) in Scott's version, but also to be found in Gödel's notebooks, blocks the alleged inconsistency.



REFERENCES

Robert M. Adams

[1971] "The Logical Structure of Anselm's Arguments," *Philosophical Review*, vol. 80, no. 1, pp. 28-54.

[1995] "Introduction Note to *1970 [Ontological Proof]" in Feferman *et al.* [1995], GCW-III.

C. Anthony Anderson

[1990], "Some Emendations of Gödel's Ontological Proof," *Faith and Philosophy* vol. 7, pp. 291–303.

C. Anthony Anderson and M. Gettings

[1996] "Gödel's ontological proof revisited," *Gödel '96: Logical Foundations of Mathematics, Computer Science, and Physics: Lecture Notes in Logic* 6, Springer 1996.

C. Anthony Anderson and Mikhail Zeleny (eds.)

[2001] *Logic, Meaning and Computation: Essays in Memory of Alonzo Church*. Kluwer Academic, Netherlands.

Matthias Baaz, Christos Papadimitriou, Hillary Putnam, Dana Scott, Charles Harper, Jr. (eds.)

[2004], *Kurt Gödel and the Foundations of Mathematics: Horizons of Truth* (Cambridge University Press: New York).

K. Jon Barwise and John Etchemendy

[1989] *The Liar*, Oxford University Press: New York, New York.

Christoph Benzmüller, Bruno Woltzenlogel Paleo

[2013] "Formalization, Mechanization and Automation of Gödel's Proof of God's Existence", http://arxiv.org/abs/1308.4526

[2016] "The Inconsistency in Gödel's Ontological Argument: A Success Story of AI in Metaphysics", *Proceedings of the Twenty-Fifth Joint Conference on Artificial Intelligence* (IJCAI-16), 936-42.

Stephen Budiansky

[2021] *Journey to the Edge of Reason: The Life of Kurt Gödel*, W. W. Norton & Company.

Jack Bulloff, Thomas Holyoke, S. W. Hahn (eds.)

[1969, *Foundations of Mathematics: Symposium Papers Commemorating the Sixtieth Birthday of Kurt Gödel* (Springer-Verlag: New York).

John Burgess

[1984," "Basic Tense Logic" in Gabbay and Guenthner [1984], 89-113.

Lewis Carroll (Charles Dodgson)

[1895], "What the Tortoise Taught Achilles", *Mind*, vol. 4.

Gregory J. Chaitin

[2012] "*Mechanical Intelligence versus Uncomputable Creativity"* in *Cooper et al*. [2012], 55.

Alonzo Church

[1935] An unsolvable problem of elementary number theory (abstract), *Bull. Am. Math. Soc.* **41**, 332-333.

[1936] An unsolvable problem of elementary number theory, *Am. J. Math.* **58**, 345-363.

[1936*a*] A note on the *Entscheidungsproblem*, *J. Symb. Log.* **1**, 40-41, Correction, *ibid*., 101-102.

[1937] Review of Turing [1936/7], *J. Symb. Log.* **2,** 42, reprinted in Cooper and Leeuwen [2014], 209.

[1940] On the concept of random sequence, *Bull. Am. Math. Soc.* **46**, 130-135.

[1944] *Introduction to Mathematical Logic, vol. I*, Annals of Mathematics Studies, revised in 1956 Princeton University Press, reprinted in 1996 in the Princeton Landmarks in Mathematics series.

John H. Conway and Joseph Shipman
[2014] "Extreme proofs I: The irrationality of √ 2", reprinted in *The Best Writing on Mathematics 2014, (ed.)* Mircea Pitici, Princeton University Press, 216-27, from *The Mathematical Intelligencer* 35, no. 3 (2013): 2-7.

Barry S. Cooper and Jan van Leeuwen (eds.)
[2013] *Alan Turing: His Work and Impact*, Elsevier: Amsterdam.

B. Jack Copeland
[2013] "Turing and the Physics of the Mind", in Cooper and Leeuwen [2013], 651-9.

B. Jack Copeland, Carl Posy and Oron Shagrir (eds.)
[2013] *Computability: Turing, Gödel, Church and Beyond*, MIT Press: Cambridge, Mass.

Martin Davis
[1981] "What is Computation?", Lynn Arthur Steen (ed.), *Mathematics Today,* Springer Verlag: New York.
[1965] *The Undecidable: Basic Papers on Undecidable Basic Propositions, Unsolvable propositions and Unsolvable Problems and Computable Functions*, Raven Books.

John W. Dawson, Jr.
[1997] *Logical Dilemmas: the life and work of Kurt Gödel* (A. K. Peters, Ltd.: Wellesley, MA).

Albert Einstein
[1905] *Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt*, *Annalen der Physik*, 17 (6): 132–148.
[1915] *Die Feldgleichungen der Gravitation, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*, (part 2), 844–847.

Anita Burdman Feferman and Solomon Feferman
[2008], *Alfred Tarski: Life and Logic* (Cambridge University Press).

Solomon Feferman, J. Dawson, S. Kleene, G. Moore, R. Solovay, J. van Heijenoort, eds.
[1986] *Collected Works. Volume I, Publications 1929-1936*, OUP, 1986. [GCW-I]
[1990] *Collected Works. Volume II, Publications 1938-1974*, OUP, 1990. [GCW-II]

Solomon Feferman, J. Dawson, W, Goldfarb, C. Parsons, R. Solovay, eds
[1993] *Collected Works. Volume III, Unpublished essay*, OUP, 1993. [GCW-III]

Solomon Feferman, J. Dawson, W, Goldfarb, C. Parsons, W. Sieg, eds.
[2003] *Collected Works. Volume. V, Correspondence, H-Z,* OUP, 2003. [GCW-V]
[2003] *Collected Works. Volume. IV, Correspondence, A-G*, OUP, 2003. [GCW-IV]

Melvin Fitting
[2002] *Types, Tableaus, and Gödel's God* (Dordrecht, North Holland: Kluwer Academic).

D. Gabbay and F. Guenthner (eds.)
[1984] *Handbook of Philosophical Logic, Vol. II, Extensions of Classical Logic*, (D. Reidel: Dordrecht).

George Gamow
[1946, "Rotating universe?", *Nature*, 158, 549.

Gerhard Gentzen
[1936], *Die Widerspruchsfreiheit der reinen Zahlentheorie*, *Mathematische Annalen* 112: 493–565.

Kurt Gödel

[1930] *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, *Monatshefte für Mathematik und Physik* 37, 349-60, in GCW-I, 102-23.

[1931] *Über formal unentscheidbare Sätze der* Principia Mathematica *und verwandter Systeme, I*, *Monatshefte für Mathematik und Physik* 38, in GCW-I, 145-95.

[1933*f*] "On Intuitionistic propositional calculus," in GCW-I, 222–5.

[*1933*o*] "The present situation in the foundations of mathematics," in GCW-III, 45-53.

[*1946/9-B2], "Some observations about the relationship between theory of relativity and Kantian philosophy," GCW-III, 230-59.

[1949*a*] "A remark about the relationship between relativity theory and idealistic philosophy," in *Albert Einstein: Philosopher-Scientist*, ed. P. A. Schilpp, 555-62 (Open Court: La Salle, Illinois) reprinted in GCW-II, 202-7.

[*1949*b*] "Lecture on rotating universes," in GCW-III, 269-87.

[1949*c*] "An example of a new type of cosmological solution of Einstein's field equations of gravitation," *Review of Modern Physics*, 21, 447-50, reprinted in GCG-II, 190-98.

[1952] "Rotating universes in general relativity theory," in the *Proceedings of the International Congress of Mathematicians*, 1, Graves *et al*. (eds.), 175-81, reprinted in GCW-II, 208-16.

[*1970} "Ontological proof", in GCW-III, 403-4.

Petr Hájek

[2006] "Gödel's Ontological Proof and Its Variants" in Bass *et al.* [2006], 307-320.

Charles Hartshorne

[1962] *The Logic of Perfection* (LaSalle, Il.: Open Court).

Stephen Hawking

[2002] "Chronological protection," in *The Future of Spacetime*, S. W. Hawking, *et al.*, (Norton, New York), 87–90.

Leon Henkin

[1952] "A Problem Concerning Provability," *Journal of Symbolic Logic* 17: 160.

Stephen C. Kleene

[1936] l-definability and recursiveness, *Duke Math. J.* **2**, 340-353.

Stephen C. Kleene and Barkley J. Rosser

[1935] The inconsistency of certain formal logics, *Ann. Math.* **36**, 630-636.

Donald Kalish, Richard Montague, and Gary Mar

[1980, 2000] *Logic: Techniques of Formal Reasoning (2nd edition*), HBJ [1980] and OUP [2000].

Saul Kripke

[1959] "A Completeness Theorem for Modal Logic," *Journal of Symbolic Logic* 24: 1-14.

[2013] "The Church-Turing 'Thesis' as a Special Corollary of Gödel's Completeness Theorem" in Copeland, Posey, Shragir (eds.) [2013].

George Lakoff and Raphael Nunez

[2000] *Where Mathematics Comes From,* Basic Books, New York.

Martin H. Löb

[1955] "Solution of a Problem of Leon Henkin," *Journal of Symbolic Logic* 20: 115–118.

Gary Mar

[2024] "Gödel's platonist philosophy of mathematics vs. Turing's mechanistic philosophy of mind Is a Kantian Reconciliation Possible?" in the online conference *Gödel and Kant on Mathematics and Physics* sponsored by the Carl Friedrich von Weizsäcker Center, University of Tübingen, Germany (Oct. 13, 2024).

[2019] "A Brief History of (the Paradoxes) of Time from (A)ristotle to (Z)eno", Kurt Gödel Society conference *Gödel's Legacy: Does the Future Lie in the Past*? at the University of Vienna (25 July 2019).

[2019] "Gödel, Turing and Time: A Computational Philosophy of Mathematics", Pacific Division of the American Philosophical Association, summarized in *"Philosophical Insights from Computational Studies: Why Should Computational Thinking Matter to Philosophers?"* Gary Mar (ed.) in *Newsletter for the Committee on Philosophy and Computers*, vol. 19, no. (Fall), 2019.

[2015] "Hao Wang's Logical Journey," *Journal of Chinese Philosophy*, Wiley Blackwell, vol. 42, 540-61.

[2014] "Chaotic, Fuzzy, and Imaginary Liars," *Vienna Summer of Logic* July 9 – 21, 2014, Logic Colloquium: *Logic, Algebra and Truth Degrees*, 213-17.

[2007] Review of *Gödel's Theorem: An Incomplete Guide to Its Use and Abuse* by Torkel Franzén, *The Mathematical Intelligencer*, vol. 29, no. 2, 66-70

John M. E. McTaggart

[1908] The Unreality of Time" *Mind*, vo. 17 (68): 457-74.

[1927] *The Nature of Existence.*

Isaac Newton

[1689] *Philosophia Naturalis Principia Mathematica, Book I* (transl. by A. Motte, rev. by F. Cajori), University of California Press: Berkeley, 1934.

Charles Parsons

[2010] "Gödel and Philosophical Idealism", *Philosophia Mathematica*, III, 18, 166-192.

Roger Penrose

[1994] *Shadows of the Mind: A Search for the Missing Science of Consciousness* OUP.

[1989] *The Emperor's New Mind: Concerning computers, minds, and the laws of physics*, OUP.

Arthur N. Prior

[1950] "Thank goodness that's over", *Philosophy*, vol. 34, 12-17.

Wolfgang Rindler

[2006] "Gödel, Einstein, Mach, Gamow, and Lanczos: Gödel's remarkable excursion into cosmology" in Baaz *et al.*, 185-211.

Siobahn Roberts

[2015] *Genius at Play: The Curious Mind of John Horton Conway*, Bloomsbury, USA.

Bertrand Russell

[1903] *Principles of Mathematics*, W. W. Norton: New York.

[1959] *My Philosophical Development,* George Allen & Unwin: London.

[1951-69] *The Autobiography of Bertrand Russell,* 3 vols., London: George Allen & Unwin.

Bertrand Russell and Alfred North Whitehead

[1910-13] *Principia Mathematica*, 3 vols., self-published.

Alfred Tarski

[1936] "*Der wahrheitsbegriff in den formalisierten sprachen*", *Studien Philosophica*, vol. 1 (1936), 261-405 translated by J. H. Woodger as "The Concept of Truth in Formalized Languages", in *Logic*, *Semantics and Metamathematics*, J. Corcoran (ed.) (Hackett Publishing: Indianapolis, Indiana), 1983.

[1944] "The Semantic Conception of Truth: and the Foundations of Semantics", *Philosophy and Phenomenological Research,* vol. 4, no. 3 (March), 341-76.
[1969] "Truth and Proof", *Scientific American*, vol. 220, no. 6 (June), 63 – 77.

S. K. Thomason
[1972] "Semantic analysis of tense logic," *Journal of Symbolic Logic* 37: 150-8.

Alan Turing
[1952] "Can Automatic Calculating Machines Be Said to Think? BBC broadcast, 14 and 23 January, Cooper and Leeuwen [2013], 667-76.
[1951] "Intelligent Machinery: A Heretical Theory, BBC broadcast, Cooper and Leeuwen [2014], 664-66.
[1950*a*] "Can Digital Computers Think?" BBC broadcast 15 May 1950, Cooper and Leeuwen [2013], 660-63.
[1950] "Computing Machinery and Intelligence", *Mind,* vol. LIX, no. 236, October 1950, 433–460,
[1938] "Systems of Logic Based on Ordinals", Ph.D. (Princeton) under Alonzo Church, summarized in *Proceedings of the London Mathematical Society* (2) (1939), 161–228.
[1937] "Computability and λ-definability", *Journal of Symbolic Logic,* vol. 2, 153-163.
[1936/7] "On computable numbers, with an application to the *Entscheidungsproblem*", *Proceedings of the London Mathematical Society*, vol. 42 (2), 230-65, corrections *ibid*. **43** (1937) 544-46.

Johan van Bentham
[1984] "II.4. Correspondence Theory" in Gabbay and Guenthner [1984], 167-247.

John von Neumann
[1951] "Tribute to Dr. Gödel," in Bulloff, *et al.* [1969], *ix-x.*

Hao Wang
[1997] *A Logical Journey: From Gödel to Philosophy*, Cambridge, Mass: MIT Press.
[1987] *Reflections on Kurt Gödel*, Cambridge, Mass.: MIT Press.

ENDNOTES

[1] *"The world is rational".* This is the first in a list of 14 propositions under the heading "My Philosophical Outlook" in Gödel's notebooks (transcribed from Gabelsberger shorthand by Cheryl Dawson, Wang [1996], 9.4.17, 316). The last proposition is *"Religions are, for the most part, bad—but religion is not*." Three other propositions are: (6) "*There is incomparably more knowable a priori than is currently known*"; (10) *"Materialism is false"*; and (13*) "There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science".*

[2] The *American Mathematical Association* website for the Josiah Gibbs lectures notes that the first published version of Gödel's 25th Josiah Gibbs lecture in the *Gödel Collected Works,* vol. III omits the word 'philosophical' from its title. The subtitle of this third published volume is (paradoxically) "*unpublished* lectures and essays". Citations from the *Gödel Collected Works* adopt the following convention:"Some basic theorems on the foundations of mathematics and their implications" is referenced as GCW-III, (*1951), 304-323.

[3] It is customary to talk about "the Gödel's Incompleteness Theorem" [1931] but there are two Incmpeteness Theorems. Gödel's "Completeness" (*Vollständigkeit*) Theorem for first-order logic and "Incompleteness" (*unentscheidbare Sätze*) theorems for Peano arithmetic are not contradictory (as the original German expressions makes clear). Gödel proved the Completeness Theorem in 1929 only a year after the problem was clearly formulated in Hilbert and Ackermann's [1928] *Grundzüge der theoretischen Logik*. Gödel's published version of the Completeness Theorem adds the Compactness Theorem and avoids mentioning the controversy between Hl=ilbert and Brouwer. Today the Completeness Theoorem is presentd using Henkin's proof, which deploys a maximal consistency lemma due to Lindenbaum. It is this version of the Completeness Theorem that bears the closest resemblence to the maximal consistency proof of God's Leibniznian perfections in Gödel's version of the ontological proof in hhigher-order modal logic.

[4] G. H. Hardy, "The Theory of Numbers," *Nature* (Sept. 16, 1922), vol. 110, 381.

[5] In what follows, we distinguish between platonism (*lower case*) or realism about mathematical objects and truths and Platonism (*upper case*), the classical views of Plato with respect to mathematics, minds, and mysticism.
[6] Turing's initiated "experimental mathematics" in two papers. A. M. Turing, "Method for the calculation of the zeta-function," *Proc. London Math. Soc*., ser. 2, vol. 48, 1943, pp. 180-197 and A. M. Turing, "Some calculations of the Riemann zeta-function," *Proc. London Math. Soc.*
[7] Gödel [*1930], GCW-III, 50.
[8] GCW-II, 268.
[9] C. P. Snow [1969], *Variety of Men*, Penguin, 25–56. Snow had encouraged Hardy, who suffered from depression, to write his now classic essay "A Mathematician's Apology".
[10] Turing [1938], section 9, reprinted in Cooper and Leeuwen [2013].
[11] Hilbert: "The art of doing mathematics consists in finding that special case which contains all the germs of generality." Quoted in Constance Reid's biography *Hilbert* [1970], Allen & Unwin, Springer-Verlag.
[12] Tannenbaum's proof, along with other "extreme proofs", is discussed in Conway and Shipman [2014].
[13] See Conway's *On Numbers and Games* [1976]. The term "surreal numbers" was coined by Donald Knuth's *Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness* [1974]
[14] Roberts [2015], 211 recounts Conway's story about meeting Gödel: "Stan was the sort of pet or protégé of Gödel's. They had an almost son-to-father relationship. He had done various things in mathematical logic, and he wrote lots of letters to Gödel, and Gödel responded. So had the ins to Gödel, and he said, 'If you like, I'll introduce you to God'—that's what he always called him. There was a thing: all the big people in mathematical logic had vaguely religious names. Gödel was known as God, Georg Kreisel was Christ, Alonzo Church was the Church, and then Errett Bishop came a good time later, and he was the Bishop. God, Church, Christ, and the Bishop, I think that's the set. So anyway, Tennenbaum offered: 'Would you like to be introduced to God?' I said, 'Yes, of course.' You don't turn an invitation like that down."
[15] The graphic created by Miranche is from the *Wikipedia* creative commons entry on Zeno's paradoxes.
[16] See Lakoff and Nunez, *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being* [2000].
[17] Wang [1996], 9.4.17, 316.
[18] The critical discussion, and refutation, of this comment by Tarski initiated Hartry Field [1972] "Tarski's Theory of Truth", *Journal of Philosophy*, vol. 69, no. 13, 347 has generated a large literature. That Tarski's offhand remark is easily refutable suggests that it was neither central to his work on truth nor seriously intended as a philosophical claim. Tarski first met Gödel during his visit to Vienna Circle in 1930, at the invitation of Karl Menger. At that time, Gödel had already independently discovered Tarski's Indefinability of Truth Theorem [1933] but framed his theorem in terms of unprovability perhaps also to circumvent the verificationist prejudices of the Vienna Circle.
[19] I am indebted to my student May Rose Connor for pointing out that the Latin word translated "enemy" has the sense of personal enemy.
[20] Gödel, *Letter to Hao Wang*, 7 Dec. 1967, GCW-V, 396-99.
[21] The original of Turing's essay "Nature of Spirit: is at the Turing Archive, King's College, Cambridge, ref. AMT/C/29]. See also *https://oldshirburnian.org.uk/alan-turing-and-the-nature-of-spirit/*.
[22] B. Jack Copeland [2013], "Turing and the Physics of the Mind", in Cooper and Leeuwen [2013], 651-9. In his introduction to transcripts of the Turing BBC Broadcasts, Copeland comments: "[T]he feature of the broadcast that is of absolutely outstanding interest, and is the topic of this note, is Turing's brief discussion of the possibility that physical action is not always computable. Roger Penrose attributed the opposite view to Turing. He said, 'It seems likely that he [Turing] viewed physical action in general – which would include the action of a human brain – to be always reducible to some kind of Turing-machine action' (Penrose, 1994, p.21). Penrose even named this claim Turing's thesis. Yet Turing never endorsed this thesis. As 'Can Digital Computers Think?' makes clear, Turing was aware that the thesis might be false."
[23] GCW-V, 337.
[24] Gödel [1931*a*], GCW-I, 203.
[25] A mechanical procedure, in contemporary terms, is an *algorithm.* Today Gödel's incompleteness theorems are studied through the perspective of algorithmic complexity. In a letter written in 1956 (but not discovered until the 1990s), Gödel wrote to von Neumann, who was dying from cancer, hoping to give him something to contemplate other than his impending death. In this remarkable letter, Gödel discusses what is now known as the P = NP conjecture, one of the famous unsolved problems in computer science, a Clay Institute $1 million Millennium problem.

[26] GCW-I, 149.

[27] The formal system is also *essentially incomplete*, *i.e.*, one can add the undecidable Gödel sentence as a new axiom and the resulting system will have a new undecidable sentence, which is also undecidable in the original system.

[28] Gödel [*1933*o*], GCW-III, 52.

[29] Wang [1997], 320, 9.5.10.

[30] Gödel casually remarks (GCW-III, [1949a], 271): "This contradicts Mach's principle, but it does not contradict relativity theory." Although dismissed by physicists of his time, Gödel's work anticipated contemporary speculations about the physics of time travel by means if singularities, see G. F R. Ellis "Contributions of K. Gödel in Relativity and Cosmology", pp. 34 ff. in *Gödel '96 Logical Foundations of Mathematics, Computer Science and Physics: Kurt Gödel's Legacy*. Cambridge University Press [1996], Springer Verlag, ed. Petr Häjek, Kip Thorne [1994, 2014]), the latter being a *New York Times* bestseller about the physics of the Christopher Nolen film *Interstellar* [2007]).

[31] Gödel's handwritten notes for his May 7, 1949. lecture at the IAS (GCW-III, 274).

[32] One axiom can be derived from the other by reversing the conditional and reversing the temporal operators.

[33] A linguistic motivation would be to combine modal tense logic with other features such as *counterfactuals*, modality, and Reichenbach's [1947] *analysis of tense using three points in time*: 'E' (the event), 'R' (a point of reference) and 'S' (point of speech) and two ordering relations to give an analysis of sentences such as these: "If I *had asked* Rosemary to marry me when we first met, she would have accepted. If we *had married* back then, we *would have now been* married 28 years. Fortunately, or unfortunately, you can't change the past." Another reason concerns computer science and dynamic logic. "Temporal operators have been used to express such properties of programs a termination, correctness, safety, deadlock freedom, clean behavior, data integrity, accessibility, responsiveness, and fair scheduling" (see, Burgess [1984], 95).

[34] Consider a tensed theory of time in which the *present* and *past* are *real* but in which the future is *not yet real*. For any two temporal points A and B on a closed loop such A is present and therefore real, but B exists in both A's past and future and so is both real and not-real. According to *presentism* only the present is real. Assuming that A is the present moment and therefore real, in a closed time-like loop A would also be in its own past and future and hence also unreal.

[35] See van Bentham [1987], 198, 223. Löb's axiom requires *well-foundedness*, *i.e.*, that there are no infinitely descending chains. The McKinsey axiom requires reflexive endpoints. But these two properties are inconsistent on transitive frames since any reflexive point implies the existence of an infinitely descending chain, which is forbidden by well-foundedness. However, the tense logic is consistent. Consider the collection of all finite and co-finite (*i.e.*, a set whose complement is finite) subsets of natural numbers. This collection will have all finite sets and their complements, but not subsets such as the set of all even numbers, or the set of prime numbers. Let the relative possibility relation on the set of finite and co-finite sets be proper subset relation. The Löb Axiom holds because the frame is transitive and well-founded. The McKinsey Axiom also holds since its antecedent says that if any formula denotes a co-finite set, and this set will have a future stage that *stabilizes*, i.e., a set that contains all greater natural numbers. The empty set and the set of all natural numbers constitute non-reflexive "endpoints." Thomason ([1972], 153) speculates that the McKinsey Axiom has a "reasonable intuitive meaning" related to the Second Law of Thermodynamics since it says that "each proposition eventually ceases changing its truth value with time."

[36] Schwinger went on to win the Nobel Prize in Physics [1965] jointly with Richard Feynman and Shinichiro Tomonaga for work on Quantum Electrodynamics.

[37] Beginning in 1992, Stephen Hawking postulated "Time Cops" (alluding to Isaac Asimov's *The End of Eternity*) to prevent the appearance of closed time-like curves but subsequently dropped the proposal as *ad hoc*.[37] The geometry of the multiverse of possible spacetimes in the Theory of Relativity is a subject of current research. For example, Chaitin, *et al.* [2012, 124]: *The "typical" spacetime is exotic, without global time, and if properly axiomatized in set theory with Martin's axiom, it is set-theoretically generic.*

[38] Gödel in Wang [1996], 9.5.6, 320.

[39] Wang [1997], 176, 5.4.19 reports Gödel's views: "Positivism is generally not fruitful in scientific research although it may have been valuable in the discovery of the special theory of relativity. Generally speaking, right ideas are fruitful. Positivism is pedagogically better for the special theory of relativity."

[40] Wang [1987], 16, 19, 21.

[41] Dawson [1997], 307.

[42] The Burke Grandjean questionnaire was first reported in Wang [1987] and appears in GCW-IV, 441ff.

[43] I asked Dana Scott at the 2014 *Vienna Summer of Logic* about it felt about being entrusted with Gödel's ontological discovery. Scott lamented having had been the recipient of a never-ending stream of correspondence from religious admirers, and atheistic detractors, who had little appreciation of the logical intricacies of Gödel's ontological discovery.

[44] Fitting ([2002], 115): "Perhaps Gödel had in mind something like the notion of a maximal consistent set of formulas, familiar from the Lindenbaum/Henkin approach to proving classical completeness."

[45] See Hartshorne [1962], which was praised by Gödel, Adams [1971], [1997], Hájek [2006], and Mar [1996].

[46] See van Bentham [1987], 176.

[47] Wang [1997], 293, 9.1.16.

[48] Wang [1996], Chapter 9, "Gödel's Approach to Philosophy", 9.5.6, 320.

[49] Gödel, *Philosophical Notebooks*, GCW-III, 431.

[50] Wang [1996], 9.4.18, 116.

[51] Wang [1996], 4.4.18, 151.

[52] Gödel's Letter to his mother Marianne, dated 23 July 1964, GCW-IV, 428-31 *"If the world [Welt] is rationally constructed and has meaning, then there must be such a thing [as an afterlife], for what sense would there be in creating a being (man), which has such a wide realm of possibilities for its own development and for relationships to others, and then not allowing it to realize even a thousandth of those [possibilities]? That would be almost like someone laying, with the greatest effort and expense, the foundations for a house, and then letting it all go to seed again."*