How much Time does a logical Inference take? – An Essay on the Evolution of Knowledge

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Abstract

Since ancient times, philosophers and physicists have studied the close connection between the phenomena of physical causation and time. This paper now shifts the focus into an epistemological setting and analyses the connection between logical causation and time, thus underlining an analogy between physical causality on the one hand side and logical inference on the other. The question, how much time is needed for an inference, is addressed from three different angles: in connection with Kripke models for intuitionistic propositional logic, in the light of algorithmical dynamics, and finally in parallel to the phenomenon of biological evolution, now considering an evolution of knowledge. The emerging overall picture is somewhat diffuse, and certainly, it cannot finally answer the question, how much time is actually needed by a logical inference. However, the formal investigations seem to hint at some fundamental inconsistencies in our intuitive concept of time, a result which appears to be in full accordance with Gödel's proof for the non-existence of time.

1 Introduction

The *Kurt Gödel Award 2021*, presented by the *Kurt Gödel Circle of Friends Berlin*, focuses on the question what it "mean[s] for our world view if, according to Gödel, we also assume the non-existence of time." In a physical world, the concept of time—or as one should maybe say more carefully: the psychological illusion of time—has always been closely connected to the concept of *causation*. And while Gödel himself regarded causation and time as two fundamental concepts in philosophy and metaphysics (see Figure 1),[1] the close interwovenness between the two has already been stressed since ancient times, when Aristotle, both in his *Physics* and *Metaphysics*[2] addressed causality in his account on the "Four Causes."[3] And whereas all four causes can be regarded as an *explanation*

[1]Kurt Gödel Papers, Box 11b, Folder 15, item accession 060168, on deposit with the Manuscripts Division, Department of Rare Books and Special Collections, Princeton University Library. Used with permission of the Institute for Advanced Study. Unpublished Copyright Institute for Advanced Study. All rights reserved. Transcriptions and translations by the author.

[2]*Physics* II 3 and *Metaphysics* V 2

[3]For detailed information about Aristotle's Four Causes, see *Falcon* (2019).

and an answer to the central question *why* things are as they are, it is Aristotle's third cause, the *efficient cause*, which, in a straightforward manner, underlines the temporal element of the concept of causality.

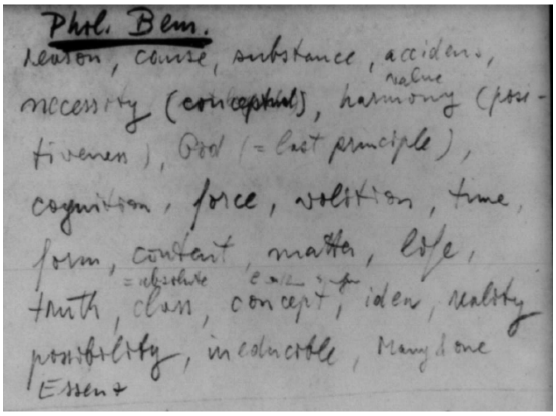

Figure 1: An undated list, written by Gödel, of fundamental philosophical concepts including *cause* and *time*. Note that *cause* is immediately preceeded by the concept *reason*. Additionally, *matter* and *form* remind of Aristotle's first and second cause.

In this paper, we now transfer the focus, which is usually seen in the connection between *physical* causation and time, into an epistemological setting, concentrating on the connection between logical inferences and the phenomenon of time. These logical inferences, which will be represented by formal rules, play a central role in the evolution of our knowledge, and as the term 'evolution' suggests, they should be closely connected to the concept of time, or at least—as mentioned before—to a psychological illusion of something we would call 'time.' Concerning the evolution of mathematical knowledge, Gödel writes in his notebook *MaxPhil IX* (p. 45):[4]

> Bem. (Phil): Wenn man die Objekte der Mathematik als durch den Geist konstruiert ansieht, bringt man notwendig ein zeitliches Element herein. Sie existieren erst nach der Konstruktion [...].
>
> [Remark (philosophy): If one regards the objects of mathematics as constructed by the mind, one necessarily brings in a temporal element. They exist only after the construction [...].]

If one agrees that a mental construction of mathematical knowledge is always based on logical inferences, then there appears to exist a striking analogy between Aristotle's *effective* physical cause on the one hand side and an epistemological cause or 'reason' on the other side. The aim of this paper now is an analysis of the link between logical reasoning and the concept of time, thus supporting a remark in Gödel's notebook *MaxPhil IV*,[5] which clearly identifies "Zeit" (time) as one of the fundamental psychological (as well as physical) concepts. Gödel writes (p. 251):

[4]Kurt Gödel Papers, Box 6b, Folder 69, item accession 030095. The book was written between November 1942 and March 1943.

[5]Kurt Gödel Papers, Box 6b, Folder 67, item accession 030090. This book was written between May 1941 and April 1942.

> Fra. (Phil.): 1. Gibt es auch eine "reine" Psychologie, welche a priori ist und in welche die inneren Wahrnehmungen eingeordnet werden, ebenso wie eine reine Physik (Raum-Zeit-Lehre)?
>
> [Question (philosophy): 1. Is there also a "pure" psychology, which is a priori and into which the inner perceptions are embedded, as well as a pure physics (space-time theory)?]

And as a footnote he adds:

> Begriffe: Zeit, actus, Erinnerung, Wahrnehmung
>
> [Concepts: time, actus, memory, perception]

As far as the title of this paper is concerned, we will certainly not be able to give a definite answer to the question how much time a logical inference takes. However, we will use the opportunity to shed light upon this question from at least three distinct angles, regarding this investigation as a collection of evidence, sometimes in favour of, sometimes against the just quoted "necessary temporal element." We postpone an answer to the question, what the non-existence of time would mean to our worldview, to our conclusion at the end of this paper.

In the course of this paper, we proceed as follows. In Section 2 we lay the basis for our discussion by fixing a set of rules of inference. In a system of *Natural Deduction*, these rules define an intuitionistic propositional logic, and we argue why this kind of logic is appropriate for our investigation. Section 3 analyses what are known as *Kripke models*, which, from the very beginning, have always been closely associated to the phenomenon of time. Section 4 then focuses on the close connection between rules of inference on the one hand side and algorithms and their inherent dynamics on the other, established by the *Curry-Howard-correspondence*. Finally, we sketch an analogy, mentioned by Gödel in unpublished notes, between biological evolution and the epistemical evolution of knowledge.

2 Rules of Inference

In this section, we introduce the rules of inference which form the basis for our discussion in the sections to come. These rules, presented in Figure 2, constitute a proof system called *intuitionistic natural deduction* (**NJ**), in which trees represent formal proofs of a *judgement* $\Gamma \vdash \varphi$, which always is the root of its proof tree. The leaves of the tree are axioms (Ax), while the inner nodes follow the construction scheme defined by the given rules. Note that the actual logical inference is always read from the top to the bottom of a rule, thus interpreting the rules in a *forward* direction. Each judgement $\Gamma \vdash \varphi$ (read "Γ proves φ") consists of a set Γ of propositional formulas, which can be interpreted as a logical environment, and a single formula φ, which holds in this very environment. A formula φ is *provable* in our logical system if $\vdash \varphi$ is the root of a proof tree.[6] For more details on intuitionistic natural deduction and examples of proof trees, see e.g. *Sørensen & Urzyczyn* (2006).

[6] In this case, the environment (i.e. the left hand side of the judgement) is taken to be the empty set.

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ (\mathrm{Ax})$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\ (\to \mathrm{I})$$

$$\frac{\Gamma \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\ (\to \mathrm{E})$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \land \psi}\ (\land \mathrm{I})$$

$$\frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \varphi}\ (\land \mathrm{E}) \qquad \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \psi}\ (\land \mathrm{E})$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \lor \psi}\ (\lor \mathrm{I}) \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \lor \psi}\ (\lor \mathrm{I})$$

$$\frac{\Gamma \vdash \varphi \lor \psi \qquad \Gamma, \varphi \vdash \tau \qquad \Gamma, \psi \vdash \tau}{\Gamma \vdash \tau}\ (\lor \mathrm{E})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ (\bot \mathrm{E})$$

Figure 2: The rules of inference for the intuitionistic natural deduction calculus **NJ**.

Before we start our discussion about the central question how much time is needed for a single logical inference, it should be worth to explain why we concentrate on an *intuitionistic* setting, thus neglecting a rule of *double negation elimination* and closely related rules and axioms like the law of the *excluded middle*.

First, the considered phenomena naturally call for an intuitionistic setting, either because the structures in question naturally constitute models for an intuitionistic logic (Section 3), or because we interpret logical formulas as simple types of certain algorithms represented by terms of combinatory logic. These types always correspond to theorems of an intuitionistic logic (Sections 4 and 5).

Secondly, it is Gödel himself who leads us into an intuitionistic direction as soon as propositions are regarded as a piece of knowledge. In his notebook *Max-Phil V* (p. 292) he writes about L. E. J. Brouwer's intuitionistic interpretation of mathematics:[7]

> Bem. (Gr.): In der Brouwerschen Interpret. der Mathematik werden gar nicht die mathematischen Sätze P interpretiert, sondern die Sätze "ich weiß P" (oder in einer schwächeren Form: P ist beweis-

[7]Kurt Gödel Papers, Box 6b, Folder 67, item accession 030091. The book was started in May 1942.

bar).

[Remark (foundations): In Brouwer's interpretation of mathematics, not the mathematical propositions P are interpreted, but the propositions "I know P" (or in a weaker form: P is provable).]

Therefore, following Brouwer, the application of a rule of inference represents an evolution of knowledge.[8] Taking rule ($\rightarrow$ E), i.e. *modus ponens*, as an example, the knowledge of both φ and the implication $\varphi \rightarrow \psi$ in the same logical environment inevitably leads to the knowledge of ψ within this environment. We will see in the following section that, apparently, this is by no means a matter of course.

Gödel's psychological view upon intuitionismus is also expressed in a remark to be found in his notebook *MaxPhil III* (p. 149), where he states:[9]

> Bem.: Nächstes Ziel für Lekt. & Arb. Unmath. sollte sein: die Grundbegriffe der Psychol. in Ordnung bringen (derart, dass man alle beschreibt und zumindest die "möglichen" Gesetze sieht, analog zu den kinematischen und Kraftbegriffen in der Physik). Rechtfertigung dafür:
>
> 1.) Anwendungen für Grundlagen (Int. ist eine schematisierte Psych.)
> 2.) Günstige Wirkung auf die Klarheit meines Denkens, die Arbeitseinteilung, Sprachbeherrschung, Arbeits-Max. ganz im Allgemeinen
> 3.) Das ist wahrscheinlich eine Voraussetzung und ein Weg zur Metaphysik und zu einer "Weltanschauung" zu kommen. Und zwar solltest du es systematisch tun.
>
> [Remark: The next goal for reading and working non-math. should be: to put the basic concepts of psychology in order (such that one describes all of them and sees at least the "possible" laws, analogous to the kinematic and force notions in physics.) Justification:
>
> 1.) Applications for foundations (intuitionism is a schematized psychology).
> 2.) Positive effect on the clarity of my thinking, organization of work, mastery of language, working maximes in general.
> 3.) This is probably a prerequisite and a way to come to metaphysics and to a "worldview." And you should do it systematically.]

[8] Gödel's claim, that intuitionistic logic is closely connected to the notion of knowledge, is also strongly supported by the Gödel-McKinsey-Tarski translation (*Gödel*, 1933), which translates intuitionistic propositions into a (classical) modal logic **S4**, in which the usual provability operator may now be interpreted as a knowledge operator. Details about the modal language of knowledge can be found, for example, in *van Benthem* (2010, Chapter 12).

[9] Kurt Gödel Papers, Box 6b, Folder 66, item accession 030089.

3 Time and the Kripke Model

As a basis for the discussion to follow, we first briefly define the notion of a *Kripke model*, which—in the case of intuitionistic logic—was first introduced in *Kripke* (1965).

A Kripke model is a triple $(W, \leq, \Vdash)$, where W is a non-empty set of *possible worlds*, $\leq$ is a partial order on W, and $\Vdash$ (read "forces") is a relation between W and the set of propositional variables, which, as a starting point, assigns atomic propositions to possible worlds, and for worlds w, w' and propositions p satisfies the *monotonicity* condition

$$\text{If } w \leq w' \text{ and } w \Vdash p \text{ then } w' \Vdash p. \tag{1}$$

From the very beginning, Kripke regarded the relation $\leq$ as an 'earlier-than'-relation. In *Kripke* (1965, p. 98), he writes under the heading "Intutive interpretation":

> We intend the nodes **H** to represent points in time (or "evidential situations"), at which we may have various pieces of information.[10]

And *Mints* (2000, p. 47) writes in his introduction to Kripke models:

> The semantics for intuitionistic logic described in the following [i.e. Kripke models] reflects a more dynamic approach: Our current knowledge about the truth of statements can improve. Some statements whose truth status was previously indeterminate can be established as true. The value *true* corresponds to firmly established truth that is preserved with the advancement of knowledge, and the value *false* corresponds to "not yet true".

Note how the idea of a "firmly established truth" corresponds to monotonicity as given in (1). An example of a Kripke model is depicted in Figure 3.

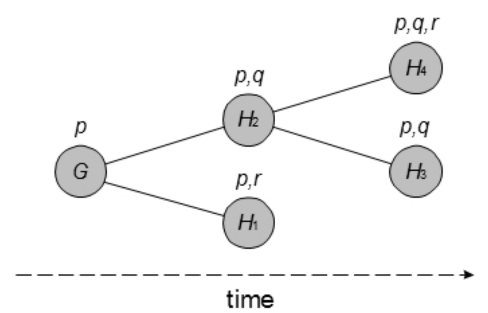

Figure 3: An example Kripke model. (The model is the one given in *Kripke* (1965), the timeline being added.)

[10] Here, **H** denotes some possible world. The "pieces of information" are represented by the propositional variables.

Finally, the forcing-relation $\Vdash$ is extended to include absurdity $\perp$ and compound logical statements which have been constructed using the usual connectives $\wedge, \vee$ and $\rightarrow$, denoting conjunction, disjunction, and implication, respectively.

$$w \Vdash \perp \text{ never holds,} \tag{2}$$
$$w \Vdash (\varphi \wedge \psi) \text{ iff } w \Vdash \varphi \text{ and } w \Vdash \psi, \tag{3}$$
$$w \Vdash (\varphi \vee \psi) \text{ iff } w \Vdash \varphi \text{ or } w \Vdash \psi, \tag{4}$$
$$w \Vdash (\varphi \rightarrow \psi) \text{ iff from } w' \Vdash \varphi \text{ follows } w' \Vdash \psi \text{ for all } w' \text{ with } w \leq w'. \tag{5}$$

If negation $\neg$ is understood as an abbreviation, i.e. $\neg\varphi \equiv (\varphi \rightarrow \perp)$, then we can also add

$$w \Vdash \neg\varphi \text{ iff } w' \Vdash \varphi \text{ does not hold for any } w' \text{ with } w \leq w'. \tag{6}$$

For the model in Figure 3, we have, for example, $H_2 \Vdash (p \wedge q)$ and $G \Vdash (r \rightarrow p)$. The reader should also note that neither $G \Vdash q$ nor $G \Vdash \neg q$ are valid in this model.

We now turn to the question, how much time a rule of inference takes in a Kripke model. And considering conjunction and disjunction first, it turns out that no time is taken at all. As the stipulations (3) and (4) are restricted to a single world or—as we should say—to a single moment in time, the knowledge of the pieces of information p and q, for example, *immediately* leads to the knowledge of $p \wedge q$. Thus, the rules ($\wedge$ I) and ($\vee$ I) seem to loose the necessity to be read in a forward direction: The knowledge of p, q, and $p \wedge q$ exists at the same time, and it thus certainly becomes more difficult to regard p and q as the *reasons* or the *premisses* for the information $p \wedge q$.

The situation becomes more complex if implication is considered in (5) and in the special case of (6). An interpretation of both rule ($\rightarrow$ I) and rule ($\rightarrow$ E) (*modus ponens*) becomes much more complicated, of the former because the logical environment would have to include present and future situations, of the latter because the information ψ is no longer *caused* or *preceded* by φ and $\varphi \rightarrow \psi$; rather, both φ and ψ now appear to be *pre*conditions for the implication $\varphi \rightarrow \psi$, thus turning the rule 'upside down.' Time seems to collapse.[11]

Gödel clearly appears to have anticipated these problems. In unpublished (undated) notes[12] concerning his 1951 Gibbs lecture (*Gödel*, 1951), he writes:

> wissen = mit Recht in jedem beliebigen Grad davon überzeugt sein (insbesondere also mit Recht beschließen, es unter keinen Umständen zu revidieren)
>
> Theorem: Ich weiß p. $\supset$ Ich bin in unmittelbarem Kontakt mit den Gegenständen der Aussage p.
>
> [to know = to be rightly convinced of it to any degree (in particular, to decide rightly not to revise it under any circumstances).

[11] These problems seem to be supported by the fact that, in Łukasiewicz's many-valued logic, the *epistemological compatibility* of conjunction and implication, expressed by the inequality $p \wedge (p \rightarrow q) \leq q$, is not valid. On the other hand, we have $p \wedge q \leq p \rightarrow q$. For details, see *Lethen* (2021).

[12] Kurt Gödel Papers, Box 12, Folder 43, item accession 060573.

> Theorem: I know p. $\supset$ I am in direct contact with the objects of the statement p.]

And although his 'definition' of the term "knowledge" (wissen) is in perfect accordance with the monotonicity of the Kripke model, the "direct contact" (unmittelbarer Kontakt) appears to be highly problematic, as—in the case of implicational statements and negations—the future is concerned as well as the present moment. Apparently, Gödel was well aware of this fact. Addressing negation, he adds a footnote to his 'definition' of knowledge, which reads:

> Weil sie Erkenntnis möglich machen, also sicherlich ihre Negation niemals anzunehmen wäre, denn das hieße erkennen wollen und gleichzeitig seine Möglichkeit zu negieren. (Für Wissen in diesem Sinne gilt das Th. nicht.) Aber woher weiß ich diese Implic.? (Zu kompliziert für eine direkte Schau. Außerdem gar nicht evident, denn ein Umlernen vielleicht möglich, und der Satz betrifft eine sehr allgemeine Aussage über unseren Erkenntnis-*Appar.*, der gar nicht mit einem Blick zu übersehen [ist].)
>
> [Because they make knowledge possible, so certainly their negation would never be accepted, because that means to want to know and at the same time to negate its possibility. (For knowledge in this sense the theorem does not hold.) But how do I know this implication? (Too complicated for a direct observation. Moreover, not evident at all, because a relearning may be possible, and the theorem concerns a very general statement about our cognitive apparatus, which cannot be overlooked by a glance).]

As an aside, it should be interesting to note that, if these remarks have indeed been written in 1951 when the Gibbs lecture was given, Gödel would have anticipated the central elements of Kripke's models—possible worlds and the fundamental monotonicity property—by more than a decade, even if he does not explicitly mention the close connection to intuitionistic logic in his notes.[13]

Finally, we have a brief look at the possible "relearning" (Umlernen) mentioned in Gödel's notes. How is it possible to gain new information in the course of time? Kripke himself writes (*Kripke*, 1965, p. 98):[14]

> Now given a point in time **G**, there are various possibilities open for gaining further information about the propositions. [...] At point **G** (representing our present information) we have proved P. For all we know, we may remain "stuck" at **G** for an arbitrarily long time, without gaining any new information. But it is possible that we will gain enough information to "jump" to point $\mathbf{H_1}$ (in which case we have a proof of R in addition to P), or to the point $\mathbf{H_2}$ (where we get a proof of Q in addition to P), or even to the points $\mathbf{H_3}$ or $\mathbf{H_4}$.

It is interesting to note that here a "jump" in time seems to rely on "proofs" of new pieces of information, and clearly these proofs cannot be based on proper

[13]The concept of branching time had been known to Gödel at least since 1935, see *Lethen* (2021b).

[14]The reader may want to compare this quote to Figure 3.

rules of inference. Rather, "gaining further information about the propositions" seems to refer to the psychological feeling of *evidence*: As soon as a certain level of evidence has been reached, as soon as a certain threshold has been crossed, the piece of information is taken for granted within the present model. Alternatively, the new information might be a contingent fact, a revealed dogma, or a 'new' plausible axiom which, up to that point of time, had not been included in the theory, but is now added for "aestetical reasons" or "reasons of completeness." Gödel himself, addressing the notion of evidence, writes in his notebook *MaxPhil III* (p. 54):

> Bem.: Alles, was irgendwie eingesehen werden kann, ist entweder[15]
>
> 1. vollkommen klar (das, was man wissen kann),
> 2. einigermaßen klar (Ersetzungssaxiom),
> 3. plausibel, d.h. annehmbar aus aesthetischen, Vollständigkeitsgründen, etc.
>
> [Remark: Everything that can be understood somehow is either
>
> 1. perfectly clear (that what can be known),
> 2. reasonably clear (axiom of replacement),
> 3. plausible, i.e. acceptable due to aesthetical reasons or reasons of completeness.]

4 Time and Algorithmic Dynamics

Algorithms have always played a central role in mathematics, and they have always been regarded as *dynamic processes* in which clearly defined distinct steps have to be carried out *one after the other*. This dynamic view also has been preserved when the first formal concepts appeared, notably in Alan M. Turings introduction of the Turing machine in 1936, where Turing speaks of a "process" carried out in distinct "moments" (*Turing*, 1936–7, p. 231). Also, these dynamics reappear in other formal algorithmic concepts like Alonzo Church's *lambda calculus*, and in *combinatory logic*, which was introduced independently in *Schönfinkel* (1924) and *Curry* (1930), and which we will take as a basis for our considerations in this section.

The language of *untyped* combinatory logic (CL)[16] can be defined by the following simple production rule, which introduces the two *combinators* S and K as well as the notion of *application*, in which we call the left and right term the *subject* and the *object*, respectively, of the application.

$$\tau ::= \mathsf{S} \mid \mathsf{K} \mid (\tau\ \tau) \tag{7}$$

Expample CL-terms are S, (SK), and $((\mathsf{KK})(\mathsf{KS}))$, of which the last one may be simplified to $\mathsf{KK}(\mathsf{KS})$, following an *association to the left*.

Computation in CL is now reflected by two axiom schemes of *weak reduc-*

[15] The first two items are marked as "analytisch" at the right margin.

[16] We restrict ourselves to a very brief introduction here. For details, see *Hindley & Seldin* (2008) and *Bimbó* (2012).

tion,[17] which can be represented as

$$\mathsf{S}xyz \rhd xz(yz) \tag{8}$$
$$\mathsf{K}xy \rhd x, \tag{9}$$

where the meta-variables x, y, z stand for arbitrary CL-terms. We give an example reduction, in which the very last term cannot be reduced any further and thus constitutes a *normal form*.

$$\mathsf{K(SKSS)K} \rhd \mathsf{K(KS(SS))K} \rhd \mathsf{KSK} \rhd \mathsf{S}$$

The reader should note that we could also have chosen a different path of reductions, which would nevertheless have led to the same result, a property of CL known as the *Church-Rosser-property*.

$$\mathsf{K(SKSS)K} \rhd \mathsf{SKSS} \rhd \mathsf{KS(SS)} \rhd \mathsf{S}$$

In *typed* combinatory logic, *principal types* can be assigned to terms. Two axioms assign the principal types $(a \to (b \to c)) \to ((a \to b) \to (a \to c))$ and $a \to (b \to a)$ (or any other alphabetic variants) to the terms S and K, respectively. If $\varphi \to \psi$ and ϑ are the principal types of terms σ and τ, one determines a 'minimal' variable-substitution *sub* which syntactically unifies φ and ϑ. This unification then enables an application of the rule *modus ponens*, which finally leads to the conclusion $sub(\psi)$. These rules are summarised in Figure 4. An example, which finally assignes the principal type $a \to a$ to the term SKK is shown in Figure 5.[18]

$$\frac{}{\mathsf{S} : (a \to (b \to c)) \to ((a \to b) \to (a \to c))}\ (\mathrm{Ax})$$

$$\frac{}{\mathsf{K} : a \to (b \to a)}\ (\mathrm{Ax})$$

$$\frac{\sigma : \varphi \to \psi \qquad \tau : \vartheta}{(\sigma\,\tau) : sub(\psi)}\ (\mathrm{pt})$$

Figure 4: Assigning principal types to terms in CL. In rule (pt), *sub* is a 'minimal' substitution which syntactically unifies φ and ϑ.

The reader up to now unfamiliar with typing systems will have noticed that types are nothing but implicational propositions, and indeed, this close correspondence between propositions and types is often referred to as the *propositions-as-types-* or *Curry-Howard-correspondence*.[19] Note that, within this framework, the term may also be regarded as a *proof* for its principal type. Thus, the example in Figure 5 also demonstrates that SKK is a proof for the proposition

[17] As we do not consider the notion of *strong reduction* in this paper, we will simply speak of *reduction*.

[18] The reader should note that the term $\mathsf{SKK}x$ reduces to x for any CL-term x. It thus represents an identity operator with principal type $a \to a$.

[19] In the context of propositional logic, rule (pt) is often called *rule* **D** or *condensed detachment*.

$$
\dfrac{\dfrac{\mathsf{S} : (a \to (b \to c)) \to ((a \to b) \to (a \to c)) \qquad \mathsf{K} : p \to (q \to p)}{\mathsf{SK} : (p \to q) \to (p \to p)}\,(\mathrm{pt}) \qquad \mathsf{K} : a \to (b \to a)}{\mathsf{SKK} : a \to a}\,(\mathrm{pt})
$$

Figure 5: A tree assigning the principal type $a \to a$ to the CL-term SKK, using two applications of rule (pt). The substitution in the first step is $\{a \rightsquigarrow p, b \rightsquigarrow q, c \rightsquigarrow p\}$, in the second step $\{p \rightsquigarrow a, q \rightsquigarrow (b \to a)\}$.

$a \to a$. Without going into further details, we mention that the rules in Figure 4 are indeed sufficient to prove the whole implicational fragment of intuitionistic propositional logic (without the absurdity $\perp$). The details can be found in *Hindley* (1997).

Before we move on, we mention as an aside, that Gödel, in his notebook *MaxPhil X*,[20] comments on the proposition $a \to (b \to a)$, which is the principal type of the combinator K, and which is also known as the *Positive Paradox*, as follows. Note the close correspondence to the monotonicity property of the Kripke model, and to Gödel's definition of "wissen" (to know).

> Bem. (Gr.): Der psych. Sinn von $a \supset (b \supset a)$ ist:[21] Das Festhalten an einem einmal gefällten Urteil, wenn etwas Neues gefunden wird. Es besteht irgendwie die psych. Tendenz, wenn man durch eine neue Erkenntnis zur Umstoßung einer alten veranlasst wird, das irgendwie nicht als tatsächliches Zurücknehmen gelten zu lassen. ("So wurde das nicht gemeint.")
>
> Remark (foundations): The psychological sense of $a \supset (b \supset a)$ is: Holding on to a judgment once made, when something new is found. There is somehow the psychological tendency, when one is prompted by a new finding to overturn an old one, to somehow not let that count as an actual withdrawal. ("That's not how it was meant.")

We now return to our central question, how much time a rule of inference takes, this time regarded in the light of algorithmic dynamics. Starting with rule (pt), which is obviously closely related to rule ($\to$ E), *modus ponens*, one can see that it actually takes place *before* the algorithm starts running: If we regard an application of terms as nothing but a syntactical juxtaposition of the subject and the object of the application, we are rather confronted with a phenomenon which can be described as the stretching of a rubber band: Time seems to stand still. Nevertheless, the stretching does *trigger* the actual run of the corresponding algorithm.

In order to find out what kind of inference takes place while the algorithm is actually running, we first have a look at the rule of *substitution*, which has not explicitly been considered so far. This rule can be regarded as a way to specialise already proved knowledge. And while the calculi given in Figures 2 and 4 do not include the rule of substitution, it is an admissable rule in intuitionistic propo-

[20] Kurt Gödel Papers, Box 6b, Folder 70, item accession 030096. The book was written between March 1943 and January 1944.

[21] Gödel adds the comment " = 'wenn' " above the second $\supset$. ("wenn" means both "when" and "if.")

sitional logic.[22] The rule of substitution can simply be presented as follows, sub being any substitution which replaces variables by arbitrary propositions:

$$\frac{\varphi}{sub(\varphi)} \text{ for any variable substitution } sub \tag{10}$$

As an example, the substitution $\{a \rightsquigarrow (a \to b)\}$ would transform the proposition $a \to a$ into the specialised proposition $(a \to b) \to (a \to b)$.

As a single step of an algorithm is now represented by a single reduction in combinatory logic, we consider the principal types of a CL-Term and of the corresponding reduced term. Taking I as an abbreviation for the term SKK, the example term SKSI reduces to $\mathsf{KI(SI)}$ in a first step, which, in a second step, can be reduced to the term I. But while the original term SKSI has the principal type $(a \to b) \to (a \to b)$, the two following terms both have the principal type $a \to a$, of which $(a \to b) \to (a \to b)$ is a substitution instance. The situation is depicted in Figure 6. As it turns out, the development of the types shown

CL-term (algorithm):	SKSI	$\triangleright$	$\mathsf{KI(SI)}$	$\triangleright$	I
Principle type (knowledge):	$(a \to b) \to (a \to b)$		$a \to a$		$a \to a$

time

Figure 6: The example algorithm SKSI reduces to $\mathsf{KI(SI)}$ and finally to I. The knowledge, represented by the corresponding types, seems to decrease over time.

in the example is not a coincidence, and it can be shown in general that for CL-terms σ and τ with principal types φ and ψ, respectively, a reduction $\sigma \triangleright \tau$ always implies that φ is a substitution instance of ψ. Thus, knowledge seems to decrease over time while the algorithm is running: The rule of substitution (10) is turned upside down, the effect appears before the cause, again time seems to collapse.

As a closing remark for this section, we mention that the reason for the fact, that a reduction in combinatory logic changes the corresponding principal types, seems to be well hidden in rule (pt), which allows for a variable substitution in both the subject and the object of an application, heavily relying on a procedure known as (bidirectional) *unification*, which was first considered in *Robinson* (1965). Unpublished joint work with Anna Maiworm and Isabelle Sauer has recently shown that a restriction to a unidirectional unification, generally known as *pattern matching*, does indeed preserve the principal type when a CL-term is weakly reduced.[23] This seems to indicate that the subject of an application should not be given the power to alter the object in any way. We will briefly return to this phenomenon in the following section, where the object will be interpreted as a fixed *point of knowledge*.

[22] Admissibility means that any proof of a proposition which uses this rule can be replaced by a proof without it.

[23] For the difference between uni- and bidirectional unification, see *Knight* (1989).

5 Time and the Evolution of Knowledge

This section, in which we take a look at the intimate connection between biological evolution and the evolution of knowledge, has been inspired by a quote which can be found in Gödel's "Aflenz"-book on quantum mechanics.[24] Here Gödel writes in item 263:

> Es ist tatsächlich bestechend, die zweckmäßigen Umstellungen des Individuums (Gedächtnis) aus demselben Prinzip zu erklären wie die zweckmäßigen Umwandlungen der Arten (Anpassung).
>
> [It is indeed convincing to explain the purposeful transformations of the individual (memory) in terms of the same principle as the purposeful transformations of the species (adaptation).]

And in another place, he phrases the same analogy as follows:[25]

> Es ist tatsächlich unbefriedigend, die zweckmäßigen Reaktionen der einzelnen Individuen (Lernen) durch ein vollkommen anderes Prinzip zu erklären als die zweckmäßigen Reaktionen der Arten (Anpassung).
>
> [It is indeed unsatisfactory to explain the purposeful reactions of individuals (learning) by a completely different principle than the purposeful reactions of the species (adaptation).]

This very idea can already be found in Henri Bergson's writings, who puts the emphasis on the close relation between nature's creation of new species on the on hand side, and intellectual, i.e. human, creation and invention on the other side. In *Bergson* (1904), he writes:

> Si la vie est une création, nous devons nous la représenter par analogie avec les créations qu'il nous est donné d'observer, c'est-à-dire avec celles que nous accomplissons nous-mêmes.
>
> [If life is a creation, we must imagine it by analogy with the creations that we are given to observe, that is to say with those that we ourselves accomplish.]

And Bergson even goes so far as to claim that the analogy is not a mere coincidence (as cited in *Hadamard*, 1945, p. xii):

> The inventive effort which is found in all domains of life by the creation of new species has found in mankind alone the means of continuing itself by individuals on whom has been bestowed, along with intelligence, the faculty of initiative, independence and liberty.

In what follows, we will utilise Gregory Chaitin's metabiological model of evolution as presented in *Chaitin* (2012). This model has often been criticised as being far too simple and thus as not being able to display any interesting

[24] Kurt Gödel Papers, Box 6a, Folder 59, item accession 030082.

[25] Notebook *Quantenmechanik II*, Kurt Gödel Papers, Box 6b, Folder 78, item accession 030107.

behaviour.[26] Notwithstanding this critique, Chaitin's model will first enable us to bring out the just mentioned "inventive effort" in a precise and rigour way, and we will suggest combinatory logic—interpreted a as prototype programming language—as a means to actually implement Chaitin's algorithm. Secondly, the model will enable us to transfer the mechanism of the creation of new species into an epistemological setting, thus—as Gödel puts it—explaining "the purposeful transformations of the individual in terms of the same principle as the purposeful transformations of the species."

As it turns out, our formal approach leads to the insight that human creation does indeed simulate nature's—or, if the reader prefers—divine creation in a very precise sense. If this simulation is taken for granted, the mental construction of ideas should take a certain amount of time, just as the phenomenon of time has always been inevitably connected to the process of Darwinian evolution. And in fact, Gödel himself was concerned with the very question, how much time biological evolution would take. In a notebook entitled "Physik (1935)," he writes:[27]

> Biologie: Mögliche Anwendungen der Mathematik:
>
> a.) Wie viele Generationen seit Beginn des Lebens bis zum Menschen? Wie schnell muss die Entwicklung vor sich gegangen sein? (Zufällige Schwankungen wie groß?)
> b.) Anzahl der möglichen Molekülgruppierungen in einer Keimzelle daraus berechnen. Wie große Schwankungen der Eigenschaften des Tiers ruft die kleinstmögliche Schwankung der Eigenschaften der Zelle hervor?
>
> [Biology: Possible applications of mathematics:
>
> a.) How many generations from the beginning of life to human beings? How fast must the development have progressed? (How wide are the random variations?)
> b.) From this, calculate the number of possible molecule groupings in a germ cell. How much variation in the properties of the animal does the smallest possible variation in the properties of the cell cause?]

5.1 Chaitin's Model of Evolution

Evolution in Chaitin's model is an evolution of a single piece of software.[28] In order to run this software, we first fix a universal computer C, which always takes a program and some data for this program as its input. Evolution then starts with a randomly chosen program A, which, along with an empty set of data ε, is run on the computer C. The output is then interpreted as a natural number a, called the *fitness* of the program A. Next, we randomly choose a mutation M, which is nothing but another program for the computer C. M is

[26] For a critique, see for example *Siedliński* (2017) and *Ewert et al.* (2013). This is certainly not the place to give a comprehensive answer to this critique. Nevertheless, we rather prefer to have the question, whether the model is "inspiring either for computer scientists [...] or for biologists" (*Siedliński*, 2017, p. 143), be decided by the test of time.

[27] Kurt Gödel Papers, Box 6b, Folder 77, item accession 030105.

[28] Here Chaitin follows the general idea that DNA may be regarded as software.

then applied to the program A, yielding a new program A'. Again, the fitness a' of the program A' is computed, which is then compared to the current fitness a. If a' is greater than a, the program A' is taken as our newly produced 'organism,' A being discarded. Otherwise A' itself is discarded: an implementation of the survival of the fittest. Afterwards, a new random mutation M is chosen and the whole process starts again. The procedure is coded as Algorithm 1 below.

Algorithm 1 Chaitin's metabiological algorithm

1: $A \leftarrow$ random program
2: $a \leftarrow C(A, \varepsilon)$
3: **loop**
4:     $M \leftarrow$ random program
5:     $A' \leftarrow C(M, A)$
6:     $a' \leftarrow C(A', \varepsilon)$
7:     **if** $a' > a$ **then**
8:         $A \leftarrow A'$
9:         $a \leftarrow a'$
10:     **end if**
11: **end loop**

Needless to mention, we are concealing some of the details in our all-too-brief description of the process. What kind of 'software' are we talking about, what is the programming language? How do we interpret an output of the computer C, sometimes as an integer, sometimes as a new program? How do we choose a program 'at random'? But before we address at least some of these questions, we briefly mention a truly fundamental drawback of Chaitin's algorithm, which surfaces in lines 2, 5, and 6, or—in other words—whenever the computer C is evaluating its input. Here, one does not know whether the computation is going to halt or not: The computer underlies the famous *halting problem*. Thus, Chaitin's model of evolution has to rely on an *oracle* which predicts whether a program gets stuck in an endless loop. If so, a new random mutation (or random organism) will have to be chosen. Again, the question, if we encounter a divine element here, is left to the reader. In any case, the dynamics of the underlying algorithm perfectly matches the dynamics of Darwinian evolution. Without going into the details here, Chaitin's main result now states that the overall fitness would grow much slower if an algorithm simply produced random organisms (i.e. programs) until it finds a fitter one. Thus, the secret of a fast growing fitness seems to lie in the fact that the fittest organism is not discarded but 'used' and 'incorporated' in the next round of the algorithm.

Turning to the question of the programming language, it is worth mentioning that the literature is largely silent about this decision. One of the exceptions, if not the only one, is the paper *Ewert et al.* (2013), which uses the language $\mathcal{P}''$ introduced and analysed in *Böhm* (1964) and *Böhm & Jacopini* (1966). The programs of $\mathcal{P}''$ are strings built of the four symbols $R, \lambda, (,)$ only, following the production rule

$$\pi ::= R \mid \lambda \mid \pi\pi \mid (\pi) \tag{11}$$

An examble $\mathcal{P}''$ program might thus look like this:

$$RR\lambda(\lambda R(R\lambda))R$$

Similar to Turing machines, $\mathcal{P}''$ programs act on symbols on an infinite tape, which serves as the computer's memory.[29]

If the symbols $R, \lambda, (,)$ are now used as the possible tape-symbols as well, one can achieve that $\mathcal{P}''$ programs can actually be applied to other $\mathcal{P}''$ programs. However, *Ewert et al.* (2013) have not been able to reproduce Chaitin's theoretical results about a fast-growing fitness, thus concluding that "metabiology does not demonstrate successful Darwinian evolution" and that "although elegant in conception, metabiology departs from reality because it pays no attention to resource limitations." The authors, though, do not take into consideration a major drawback of the use of the language $\mathcal{P}''$ in a metabiological setting: If the currently fittest program is stored on the tape and the next random mutation starts to act on this program, the mutation really *destroys*—in a very crude manner—what has been reached so far, instead of *incorporating* and *reusing* the information gained so far. Thus, the overall fitness cannot grow any faster than in a naive algorithm which always starts from scratch.

Having identified this fundamental problem with the $\mathcal{P}''$ approach, we now propose *combinatory logic* (CL) as a suitable programming language in Chaitin's algorithm. While application of one program (a mutation) to another program (the organism) is straightforward, we now define the fitness of a program simply as the number of applications appearing in its normal form.[30] Again, we will have to rely on an oracle in order to find out if a normal form exists. An example run of Chaitin's algorithm in CL is shown in Figure 7. As the fitness and thus the number of applications in the corresponding CL-terms grows very fast indeed, only the number of iterations and the fitness itself are reproduced.

```
0:  3          23: 40          42: 187
1:  4          24: 41          43: 189
2:  7          25: 44          45: 190
3:  8          28: 48          46: 191
4:  10         30: 49          49: 193
7:  11         31: 51          50: 394
8:  13         32: 55          55: 398
10: 14         36: 58          56: 400
11: 32         37: 180         57: 804
22: 37         39: 185         58: 805
```

Figure 7: The first 30 steps of an example run of Chaitin's algorithm in CL. Whenever a fitter organism (i.e. a longer CL-term) is found, the number of the current generation and the fitness of the organism are printed.

As a comparison, Figure 8 shows the first nine steps of an algorithm which produces random CL-term and simply keeps them until a fitter one is found.

[29] The instruction R moves a read/write head one position to the right. The instruction λ moves it one position to the left, after altering the symbol on the tape in a cyclic way. Parentheses indicate a loop which is executed as long as the current symbol on the tape is not the first one in the list of possible tape symbols.

[30] A more elegant solution would consider the information content of an organism A, defined as the length of the shortest CL-term which reduces to A. As this measure is not computable, we stick to the more naive (but practical) solution.

What is missing is an application of the newly produced term to a current individuum.

```
1:  14            48: 23           74:  52
9:  19            59: 26           348: 58
10: 20            72: 42           899: 104
```

Figure 8: The first nine steps of an example run of an algorithm which simply produces random CL-terms until a fitter one is found. Again, only the number of the current generation and the fitness of the organism are printed.

The main advantages of using CL as the programming language in Chaitin's algorithm can be summarised as follows:

- The currently fittest organism is reused and incorporated by the newly generated random mutation. While it might be cancelled by the mutation in certain rare cases, it may as well be duplicated or itself applied to parts of the mutation. The implementation thus supports the theoretical results.
- The model can easily be extended to a population of several organisms, which may even interact throught mutual 'sexual application.'[31]
- Combinatory logic is a very well understood theory for which many different mathematical models exist, see *Bimbó* (2012) and the literature mentioned therein.
- By using *typed* combinatory logic, one could even avoid the use of an oracle, as typable CL-terms always have a normal form.

5.2 The Evolution of Knowledge

After this brief digression into the field of biological evolution, we will now turn to the evolution of knowlegde. The close analogy will suggest that if time is needed in order to create new species, a certain amount of time is also needed for the creation of knowledge, which—in the case of purely mental creativity—can be reduced to the phenomenon of having an *idea*. And whereas it is certainly difficult to give a precise definition of the term 'idea,' at least three components should be present in order to describe this very concept:

1. Randomness,
2. Truth,
3. Applicability.

Although we regard these components as necessary, we allow for a vague perception of these terms at this stage, and they will merely serve as a guide when constructing the formal system in question.

[31]The fact that Chaitin reduces his model to one single organism has frequently been critisised, see for example *Siedliński* (2017). However, *Bergson* (1911, Chapter 1) already argues: "Strictly speaking, there is nothing to prevent our imagining that the evolution of life might have taken place in one single individual by means of a series of transformations spread over thousands of ages. Or, instead of a single individual, any number might be supposed, succeeding each other in unilinear series."

As to (1.), it should be noted that we do not construct or compute our every-day ideas algorithmically, nor do we choose them out of a given pool in a predefined manner, at least not as a conscious act. A real idea seems to crawl up from our subconscious and—at least viewed from our conscious perspective—seems to be carrying a clear element of chance. Jacques Hadamard, in his book about the psychology of invention (*Hadamard*, 1945, pp. 29–30) puts it this way:

> It cannot be avoided that this first operation take place, to a certain extend, at random, so that the role of chance is hardly doubtful in this first step of the mental process. But we see that that intervention of chance occurs inside the unconscious: for most of these combinations—more exactly, all those which are useless—remain unknown to us.

As to (2.), every idea has to be *true* in some sense. Any idea not fulfilling this condition would, when applied to our knowledge (which is of course considered to be true, following the principle of veridicality[32]), produce nothing but nonsense.

Last but not least, an idea has to be *applicable*, as it is always an idea relative to some situation or to some knowledge, in order not the be "useless," as Hadamard puts it in the quotation just given. If one is stuck in the middle of a mathematical problem, an idea has to be directed towards the solution of that very problem. It has to be applicable to a prepared set of propositions or—as we will call it later on—of *points of knowledge*. *Hadamard* (1945, p. 32) even connects applicability with beauty:

> So there remains only Poincaré's final conclusion, viz., that to the unconscious belongs not only the complicated task of constructing the bulk of various combinations of ideas, but also the most delicate and essential one of selecting those which satisfy our sense of beauty and, consequently, are likely to be useful.

In what follows, we will represent both ideas and points of knowledge as implicational formulas of propositional intuitionistic logic. In order to meet the conditions of randomness and truth of an idea, we will repeatedly produce random proofs, which are represented by terms of (typed) combinatory logic. The actual principal type will then serve as a possible candidate for an idea. In the next step, we test if this candidate (which can be considered as a *pre-idea*) can actually be applied to the current point of knowledge. Here, application is considered an application of modus ponens, following a minimal substitution either in both pre-idea and point of knowledge, or, if one prefers, in the pre-idea only.[33]

Let us consider an example: Suppose the current point of knowledge were the simple proposition $a \to a$. In a first case, the random term might be the expression SII, where I again abbreviates the term SKK. As it turns out, this term is not typable. Thus, it is not a proof of a proposition and would simply be discarded, not even having reached the status of a pre-idea. In the next case, the random proof might be the term SI, which proves the proposition $((b \to$

[32]In epistemic logic, the *veridicality* property $K_a\varphi \to \varphi$ states that what an agent a knows is always true, truth is included in the concept of knowledge.

[33]In this connection, the reader may want to skip back to the closing remark of Section 4.

$c) \to b) \to ((b \to c) \to c)$. However, this pre-idea cannot be applied to the point of knowledge $a \to a$ and is therefore discarded as well. Finally, we consider the simple random proof S with principal type $(a \to (b \to c)) \to ((a \to b) \to (a \to c))$. This pre-idea is indeed a proper idea as it can be applied to the current point of knowledge $a \to a$, yielding the new knowledge $((b \to c) \to b) \to ((b \to c) \to c)$, which is then subjected to a new idea in the following iteration of the algorithm.

Figure 9 shows an example run of the just described algorithm. Again, only the number of the iteration as well as the size (i.e. the number of implications) of the current point of knowledge is printed. As a small modification, we only update the current point of knowlede if the new knowledge is more complex by at least ten implications.

```
0:   2          213: 449       302: 4989
163: 24         251: 1038      306: 7484
173: 37         272: 1177      324: 11521
183: 57         275: 1766      327: 18053
189: 143        276: 1905      334: 28622
194: 229        284: 2858      344: 45723
197: 277        292: 3083      346: 73393
210: 416        297: 4625      362: 89541
```

Figure 9: The first 24 steps of an example run of the algorithm which produces random ideas and applies them to the current point of knowledge. Printed are the number of the current generation and the complexity of the knowledge, measured by the number of implications. In this case, the initial point of knowledge is the proposition $a \to (b \to a)$.

Finally, in order to demonstrate the enormous speed at which an evolution of knowlege now takes places, we compare the described algorithm with a version in which the current point of knowledge is always completely ignored and not incorporated in the production of new random knowledge. The modified algorithm now simply produces random proofs and checks whether the proved proposition is greater in size than the current one. What is missing is an application of the pre-ideas to the current knowledge. Figure 10 shows the resulting behaviour.

```
0: 6         59: 22        3283: 36
4: 10        244: 24       28448: 56
8: 15        331: 25       100358: 59
```

Figure 10: The first 9 steps of an example run of the algorithm which produces random proofs and keeps the according knowledge until more complex knowledge has been found. Note the enormous difference in speed compared to the example given in Figure 9, where the current knowledge is incorporated in the process.

Basically, we have been able to show that creativity can flourish even in complete isolation, i.e. without any contact to the outside world, just by tossing a coin. As *Bergson* (1911) puts it in the introduction to his *Creative Evolution*:

> [T]he intellect has only to follow its natural movement, after the lightest possible contact with experience, in order to go from discovery to discovery, sure that experience is following behind it and will justify it invariably.

We also have been able to demonstrate that the current knowledge grows by far faster if the new ideas are applied to this knowledge. In the present context it is most important, though, to note the close analogy between natural evolution on the one hand side, represented here by Chaitin's metabiological algorithm, and the evolution of knowledge on the other side, simulated by the application of random ideas to a point of knowledge, which have both been represented by logical propositions. This analogy can be regarded on two different levels. First, from an intuitive point of view, an obvious parallel between the two given algorithms meets the eye: random mutations correspond to random ideas, organisms to points of knowledge. Application and iteration each play a central role. Secondly, on a more technical level, combinatory logic, untyped and typed, serves as a convincing tool in the actual implementation of both algorithms, representing both software and proofs, linked through the Curry-Howard-correspondence.

As we have already mentioned, time has always been inseparably connected to Darwinian evolution, and we may therefore conclude that it should also be inseparably connected to human creativity and an evolution of knowledge. *Bergson* (1911, p. 11) states:

> The universe *endures*. The more we study the nature of time, the more we shall comprehend that duration means invention, the creation of forms, the continual elaboration of the absolutely new.

6 Conclusion

In his notebook *MaxPhil V*, Gödel writes (p. 350):

> Bem. (Phil.): Es gibt zwei Methoden der Philosophie, die intuitive und die kombinat[orische]. Die erste ist sehend, die zweite blind, die erste anstrengend,[34] die zweite leicht, die erste verständnisvoll, die zweite mechanisch. Die erste hat zu tun mit dem Sinn der Sprache, die zweite mit der Sprache selbst. Die erste führt zu einem lebendigen Wissen, die zweite zu einem abstakten Wissen. (Das Richtige [ist] eine Kombin[ation] beider. Ich habe bisher die zweite vernachlässigt.) Nur für die zweite (axiomat[ische]) braucht man Papier und Bleistift. Die beiden Methoden entsprechen genau den beiden Anschauungen über die Erkenntnis, dass sie ein Wahrnehmen bzw. ein Konstruieren ist.
>
> [Remark (philosophy): There are two methods of philosophy, the intuitive and the combinatorial. The first is seeing, the second blind, the first exhausting,[35] the second easy, the first understanding, the

[34] Gödel's footnote: "erfordert Konzentration"
[35] Gödel's footnote: "requires concentration"

> second mechanical. The first has to do with the meaning of the language, the second with the language itself. The first leads to a living knowledge, the second to an abstract knowledge. (The right approach is a combination of the two. I have neglected the second so far). Only for the second (axiomatic) one needs paper and pencil. The two methods correspond exactly to the two views of knowledge, that it is <u>a perceiving and a constructing respectively.</u>]

Without doubt, we have followed the second approach in this essay, having used "paper and pencil," assited by a computer executing the "mechanical" proposed simulations. And equipped with the considerations and the formal results of the different perspectives on the connection between logical inference and (an illusion of) time, we can now finally return to the question in which way the assumption of the non-existence of time may influence our worldview. For this purpose, we have to include the human mind into our worldview and analyse how the brain and our thoughts reflect the physical world surrounding us: If a physical time does not exist, still, nature has obviously equipped man with an inner *sense* of time, a sense which helps to structure the outer physical world and its events, as well as our collections of knowledge. The—without doubt—diffuse picture which emerges from the different considerations presented in this paper may now be taken as a gentle hint at an inconsistency within this very perception of (a non-existing) time, an inconsistency which may be interpreted as an immediate consequence of the non-existence itself.

It should be worthwhile to mention that the overall situation may be easily compared to a worldview in which the infinite set, especially the universal set, does not have any 'real' existence. Still, man is apparently equipped with the ability to 'perceive' infinite sets, possibly even the universal set. Again, this ability helps us to structure both the world surrounding us as well as our mental processes. And yet, the (formal) inconsistencies within naive set theory, for instance surfacing with Russell's paradox, clearly hint at the assumed non-existence.

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References

van Benthem, J. (2010). *Modal Logic for Open Minds.* Stanford: CSLI Publications.

Bergson, H. (1904). Notice sur la vie et les œuvres de Félix Ravaisson-Mollien. In: H. Bergson (1934), *La Pensée et le Mouvant. Essais et conférences.* Paris: Alcan, pp. 253–291.

Bergson, H. (1911). *Creative Evolution.* New York: Henry Holt and Company.

Bimbó, K. (2012). *Combinatory logic: Pure, applied and typed.* London: Chapman and Hall/CRC.

Böhm C. (1964). On a family of Turing machines and the related programming language. *ICC Bulletin*, 3(3), 185–194.

Böhm C. & Jacopini, G. (1966). Flow diagrams, Turing machines and languages with only two formation rules. *Communications of the ACM*, 9(5), 366–371.

Chaitin, G. (2012). *Proving Darwin: Making Biology Mathematical.* New York: Pantheon Books.

Curry, H. B. (1930). Grundlagen der kombinatorischen Logik. *American Journal of Mathematics*, 52: 509–536, 789–834.

Ewert W., Dembski, W. A., Marks, R. J. (2013). Active Information in Metabiology. *BIO-Complexity*, 2013(4), 1–10.

Falcon, A. (2019). Aristotle on Causality. In: *The Stanford Encyclopedia of Philosophy* (Spring 2019 Edition), E. N. Zalta (ed.), https://plato.stanford.edu/archives/spr2019/entries/aristotle-causality/

Gödel, K. (1933). Eine Interpretation des intuitionistischen Aussagenkalküls. *Ergebnisse eines mathematischen Kolloquiums*, vol. 4, 39–40.

Gödel, K. (1951). Some basic theorems of the foundations of mathematics and their implications. In: S. Feferman, W. Goldfarb, Ch. Parsons, et al. (eds.), *Kurt Gödel, Collected Works. Vol. III. Unpublished Essays and Lectures*. New York: Oxford University Press, pp. 304–323.

Hadamard, J. (1945). *The Psychology of the Invention in the mathematical Field.* Princeton, Mass.: Princeton University Press.

Hindley, J. R. (1997). *Basic Simple Type Theory.* (Cambridge Tracts in Theoretical Computer Science, Vol. 42). Cambridge: Cambridge University Press.

Hindley, J. R. & Seldin, J. P. (2008). *Lambda-Calculus and Combinators – an Introduction.* Cambridge: Cambridge University Press.

Knight, K. (1989). Unification: A multidisciplinary survey. *ACM Computing Surveys (CSUR)*, 21(1), 93–124.

Kripke, S. A. (1965). Semantical analysis of intuitionistic logic I. In: J. N. Crossley & M. A. E. Dummett (eds.), *Formal Systems and Recursive Functions, Proceedings of the Eighth Logic Colloquium, Oxford, July 1963*, Studies in Logic and the Foundations of Mathematics (Vol. 40). Amsterdam: North-Holland, pp. 92–129.

Lethen, T. (2021a). Gödel on many-valued logic. *The Review of Symbolic Logic*, https://doi.org/10.1017/S1755020321000034

Lethen, T. (2021). Monads, Types, and Branching Time – Kurt Gödel's Approach Towards a Theory of the Soul. In: O. Passon & C. Benzmüller (eds.), *Wider den Reduktionismus*. Berlin: Springer Spektrum, pp. 13–24.

Mints, G. (2000). *A Short Introduction to Intuitionistic Logic*. New York: Kluwer Academic.

Robinson, J. A. (1965). A machine-oriented logic based on the resolution principle. *Journal of the ACM (JACM)*, 12(1), 23–41.

Schönfinkel, M. (1924). Über die Bausteine der mathematischen Logik. *Mathematische Annalen*, 92: 305–316.

Siedliński, R. (2017). Turing Machines and Evolution. A Critique of Gregory Chaitin's Metabiology. *Studies in Logic, Grammar and Rhetoric*, 48(1), 133–150.

Sørensen, M. H. & Urzyczyn, P. (2006). *Lectures on the Curry-Howard Isomorphism*, Studies in Logic and the Foundations of Mathematics (Vol. 149). Amsterdam: Elsevier.

Turing, A. M. (1936–7). On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society* (2) 42 (1936–7), 230–265. A correction, ibid. 43 (1937), 544–546.